

Q-KOSZUL ALGEBRAS AND THREE CONJECTURES

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We dedicate this paper to the memory of J.A. Green

ABSTRACT. In previous work, the authors introduced the notion of Q-Koszul algebras, as a tool to “model” module categories for semisimple algebraic groups over fields of large characteristics. Here we suggest the model extends to small characteristics as well. In particular, we present several conjectures in the modular representation theory of semisimple groups which these algebras inspire. They provide a new world-view of modular representation theory, potentially valid for some root systems in all characteristics. In fact, we give a non-trivial example in which $p = 2$. This paper begins a systematic study of Q-Koszul algebras, viewed as interesting objects in their own right.

1. INTRODUCTION

Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded algebra over a field k . For simplicity, assume A is finite dimensional and that k is algebraically closed. Let $A\text{-grmod}$ be the category of finite dimensional \mathbb{Z} -graded A -modules, and let $A\text{-mod}$ be the category of finite dimensional A -modules. The abelian categories $A\text{-grmod}$ and $A\text{-mod}$ each have enough projective (and injective) modules. If $M = \bigoplus_i M_i \in A\text{-grmod}$, for any integer r , $M\langle r \rangle \in A\text{-grmod}$ is defined by $M\langle r \rangle_i := M_{i-r}$. If ext^\bullet denotes the Ext-bifunctor in $A\text{-grmod}$, then, for $M, N \in A\text{-grmod}$,

$$(1.0.1) \quad \text{Ext}_A^n(M, N) = \bigoplus_{r \in \mathbb{Z}} \text{ext}_A^n(M, N\langle r \rangle), \quad \forall n \in \mathbb{N},$$

where the left hand side is computed in $A\text{-mod}$, after forgetting the gradings on M and N . One says that A is Koszul provided that each irreducible module L , when regarded as a graded A -module concentrated in grade 0, has a projective resolution $P^\bullet \rightarrow L$ in $A\text{-grmod}$ in which P^n has head which is pure of grade n . Equivalently, $\text{ext}_A^n(L, L'\langle r \rangle) \neq 0 \implies n = r$ for any two irreducible A -modules L, L' concentrated in grade 0.

Ever since the pioneering work in [3] (see also [27]), Koszul algebras have played a prominent role in representation theory. For example, [3] proved that if \mathcal{O}_0 denotes the principal block for the category \mathcal{O} of a complex semisimple Lie algebra \mathfrak{g} , then \mathcal{O}_0 is equivalent to the module category of a finite dimensional Koszul algebra A . Also, [2] and [40] show the restricted Lie algebra of a semisimple, simply connected algebraic group G in characteristic $p > 0$ is Koszul, provided that p is sufficiently large, depending on the root system. Nevertheless, the Koszul property generally fails for irreducible G -modules

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outside the Janzten region (for any p). More precisely, the finite dimensional algebras A_Γ below governing the representation theory of G are mostly not Koszul.

Let Γ be a finite set of dominant weights on G which is an ideal in the dominance order. The category of finite dimensional rational G -modules which have composition factors consisting of those irreducible modules $L(\gamma)$, $\gamma \in \Gamma$, is equivalent to the module category $A_\Gamma\text{-mod}$, where A_Γ is a finite dimensional algebra. While the algebra A_Γ is necessarily quasi-hereditary, that fact alone is not, without further structure, sufficient to understand more deeply the representation theory of G . In this spirit, the recent paper [35] introduced the notion of a “standard Q-Koszul algebra” as a potential model for the representation theory of G —albeit (at that time) for large primes and p -regular weights. In more detail, assume that Γ is restricted to consist of p -regular weights. (Thus, Γ is a finite ideal in the poset of all p -regular dominant weights.) The algebra A_Γ has a filtration by ideals (which arise from the radical series of its quantum analogue), and one can form the associated graded algebra $\tilde{\text{gr}} A_\Gamma$. Then a main result in [35] proves that the algebra $\tilde{\text{gr}} A_\Gamma$ is a standard Q-Koszul algebra, provided that p is “sufficiently large.” Here “sufficiently large” means that the Lusztig character formula is assumed to hold for G , and $p \geq 2h - 2$ is odd (with h the Coxeter number of G). Further, much of the homological algebra of A_Γ can be determined from that of $\tilde{\text{gr}} A_\Gamma$, but now assuming only that $p \geq 2h - 2$ is odd. (See the remarks after Conjecture IIa in §5 below.)

A major aim of this paper, undertaken in its final sections, is to suggest a much broader role for Q-Koszul algebras in the representation theory of G , one valid also in smaller characteristics and for singular weights. Earlier sections initiate a systematic study of Q-Koszul algebras, giving complete definitions and establishing some basic (but new) general results, not even yet observed in the large prime cases studied earlier.

In more detail, a finite dimensional, positively graded algebra A is Q-Koszul provided that the grade 0 subalgebra A_0 is quasi-hereditary (with weight poset denoted Λ). In addition, it is required that

$$(1.0.2) \quad \text{ext}_A^n(\Delta^0(\lambda), \nabla_0(\mu)\langle r \rangle) \neq 0 \implies n = r, \quad \forall \lambda, \mu \in \Lambda, n \in \mathbb{N}, r \in \mathbb{Z}$$

Here $\Delta^0(\lambda)$ (respectively, $\nabla_0(\mu)$) is the standard (respectively, costandard) module of corresponding to λ (respectively, μ) in Λ . Thus, if A_0 is semisimple, A is just a Koszul algebra. But in the situations we have in mind, A_0 is hardly ever semisimple! Q-Koszul algebras are studied in §2. One main result, given in Theorem 2.2, is that A is tight (i. e., A is generated by A_0 and A_1). Then Theorem 2.3 shows that A is a quadratic algebra (see §2 for a precise definition). This suggests the (future) project of explicitly describing A by generators and relations (for particular Q-Koszul algebras of interest in modular representation theory).

Another important result, also given in Theorem 2.3, shows that if A is Q-Koszul, then the left A_0 -module A_1 has a Δ^0 -filtration—in fact, the later Theorem 2.8 shows that the (A_0, A_0^{op}) -bimodule A_1 has a $\Delta^0 \otimes_k \Delta^{0, \text{op}}$ -filtration. For $s \geq 1$ and $r \geq 0$, let $\Omega_r(A_s)$ be the r th syzygy module of A_r . Then Lemma 2.4 shows that the A_0 -modules $\Omega_{s-1}(A_s), \Omega_s(A_s), \Omega_{s+1}(A_s), \dots$ all have Δ^0 -filtrations. Of course, this extends the result just mentioned from Theorem 2.3 since $\Omega_0(A_1) = A_1$. §2 contains a number of similar

results, often cast in the more general setting of n -Q-Koszul algebras, which are sometimes assumed to be quasi-hereditary (automatic in the standard Q-Koszul case).

Section 3 is concerned with standard Q-Koszul algebras. Suppose that a finite dimensional algebra A is positively graded and quasi-hereditary (with weight poset Λ). For $\lambda \in \Lambda$, let $\Delta(\lambda)$ and $\nabla(\lambda)$ denote the corresponding standard and costandard modules, respectively. It is known from [31] that the subalgebra A_0 of pure grade 0 is also quasi-hereditary with weight poset Λ (and standard and costandard modules denoted $\Delta^0(\lambda)$ and $\nabla_0(\lambda)$, respectively, for $\lambda \in \Lambda$). We say that A is a *standard Q-Koszul algebra* provided that

$$(1.0.3) \quad \begin{cases} \text{ext}_A^n(\Delta(\lambda), \nabla_0(\mu)\langle r \rangle) \neq 0 \implies n = r; \\ \text{ext}_A^n(\Delta^0(\mu), \nabla(\lambda)\langle r \rangle) \neq 0 \implies n = r. \end{cases} \quad \forall \lambda, \mu \in \Lambda, n \in \mathbb{N}, r \in \mathbb{Z}.$$

The main result, given in Theorem 3.3, proves that if A is standard Q-Koszul, then it is Q-Koszul. Interestingly, the result is understood conceptually from a triangulated category point of view, inspired by similar methods in [8]. Another result in this section, Corollary 3.5, draws on the work of §2 to show that the grade 2 relation module W_2 of a standard Q-Koszul algebra A has an especially nice Δ^0 -filtration. This happens in spite of the fact that its grade 2 term A_2 need not have a Δ^0 -filtration. That this can occur is a consequence of an example in a Weyl module context, due to Will Turner, and discussed in §5.

Both sections 5 and 6 treat a highly non-trivial case in which the characteristic p is small. In fact, $p = 2$. Explicitly, we consider the Schur algebra $S(5, 5)$ associated to 5-homogenous polynomial representations of $GL_5(k)$ when k has characteristic 2. In this case, *all* the weights are 2-singular, and there is no proposed analog of the Lusztig character formula for irreducible modules. Nevertheless, we prove that $\tilde{\text{gr}} S(5, 5)$ is standard Q-Koszul. (Actually, we focus on $\tilde{\text{gr}} A$ for A the “principal block” of $S(5, 5)$, leaving details beyond this case to the reader. By the principal block we mean the block containing the determinant representation.) This result takes as its starting point computer calculations done by Jon Carlson [4]. It is interesting to note that while the Schur algebra $S(5, 5)$ is quasi-hereditary, the graded algebra $\text{gr} S(5, 5)$, obtained (unlike $\tilde{\text{gr}} S(5, 5)$) from the radical series filtration of $S(5, 5)$ itself, is not quasi-hereditary. In addition, $S(5, 5)$ is not Koszul (nor is the graded algebra $\text{gr} S(5, 5)$ Koszul either). See Remark 6.1.

Section 7 discusses three natural conjectures suggested by this paper in combination with our previous work. Conjecture II proposes a generalization to small primes and singular weights of (already interesting) homological results in the large prime, p -regular cases. This conjecture does not involve graded algebras in its statement. However, it is inspired by Q-Koszul theory, which might well play a role in its proof. Conjecture I asserts that a rich supply of Q-Koszul algebras is available, while two supplementary conjectures, labeled Conjectures IIa and IIb, show the relevance of these algebras to Conjecture II. Finally, Conjecture III, motivated by Koszul algebra theory in the quantum case, provides calculations, in terms of Kazhdan-Lusztig polynomials, of numbers needed to make Conjecture II explicit. All three Conjectures I, II, III, as well as Conjectures IIa, IIb, hold for the $p = 2$

example studied in §§5,6, and collapse to known or recently proved results in the large prime, p -regular weight cases.

Algebras which are Q-Koszul in our sense are also T -Koszul in the sense of Madsen [24]. This implies that the algebra $\text{Ext}_A^\bullet(T, T)$, where T is a full tilting module for A_0 (viewed as an A -module), are again T -Koszul. As formulated in Questions 4.2, we do not know if a similar permanence holds for Q-Koszul or standard Q-Koszul algebras. As the discussion of this paper shows there are a vast number of important examples of algebras which are standard Q-Koszul. We expect to return to Questions 4.2 and other issues dealing with the product structure of their Ext-algebras in a later paper.

Another topic for further research is the speculation, sketched in the final Remarks 7.2, that the conjectures of §7 may often have explicit applications to computing Ext-groups between irreducible modules.

Part I: Q-Koszul Algebras

2. Q-KOSZUL ALGEBRAS

As above, A denotes a non-negatively graded, finite dimensional algebra. Let $\pi : A \twoheadrightarrow A_0 \cong A / \sum_{i>0} A_i$ be the quotient homomorphism. If d is an integer, let $A_{\geq d} = \sum_{i \geq d} A_i$, and define $A_{< d}$, $A_{\leq d}$, $A_{> d}$ analogously. Similar notations will be used for graded A -modules. The algebra A_0 may be regarded itself as graded (and concentrated in grade 0), and every graded A -module $M = \bigoplus_r M_r$ restricts naturally to a graded A_0 -module, as does each subspace M_r , $r \in \mathbb{Z}$.

Let $M \in A\text{-grmod}$ be concentrated in grades $\geq r$. Then there is a projective $P \in A\text{-grmod}$ which is also concentrated in grades $\geq r$ and a surjective graded homomorphism $P \twoheadrightarrow M$. (One can even assume P is a projective cover of M . See [35, Rem. 8.4] for more discussion.) Thus, the kernel of the map $P \twoheadrightarrow M$ is a graded module concentrated in grades $\geq r$. This process can be continued in the evident way to obtain a graded projective resolution of M in which each term is concentrated in grades $\geq r$. A useful consequence is that, if X (respectively, Y) is a graded A -module concentrated in grades $\geq r$ (respectively, $\leq s$), then, for any non-negative integer n ,

$$(2.0.4) \quad \text{ext}_A^n(X, Y) \neq 0 \implies r \leq s.$$

Any A_0 -module M can be regarded as a graded A -module concentrated in grade 0 by making A act on M through π . Thus, there is an exact, additive functor

$$(2.0.5) \quad i_* : A_0\text{-mod} \longrightarrow A\text{-grmod}.$$

Usually, i_*M is denoted simply as M again. Of course, given A_0 -modules X, Y , this induces a linear map (still denoted i_*)

$$(2.0.6) \quad i_* : \text{Ext}_{A_0}^r(X, Y) \rightarrow \text{ext}_A^r(X, Y), \quad \forall r \geq 0.$$

It is clear that, for $r = 0, 1$, this map is an isomorphism.

Every irreducible graded A -module has the form $L\langle m \rangle$, $m \in \mathbb{Z}$, where L is an irreducible A -module concentrated in grade 0. In fact, the irreducible A -modules concentrated in grade 0 identify with the irreducible A_0 -modules. (However, we do *not* assume that A_0 is semisimple.) Let Λ be a fixed set indexing the distinct isomorphism classes of irreducible A_0 -modules.

Suppose that, in addition, A_0 is a quasi-hereditary algebra, defined by a poset structure \leq on Λ . Thus, A_0 has standard (respectively, costandard, irreducible) modules $\Delta^0(\lambda)$ (respectively, $\nabla_0(\lambda)$, $L(\lambda)$), $\lambda \in \Lambda$, satisfying the usual axioms for a highest weight category; see [6].

The previous paragraph is summarized by saying (as a definition) that A is a *0-Q-Koszul algebra*. More generally, for $n \geq 0$, A is an *n-Q-Koszul algebra* provided that A_0 is quasi-hereditary as above, and, for all $\lambda, \mu \in \Lambda$, and all $j \in \mathbb{Z}$, if $0 < i \leq n$, then

$$(2.0.7) \quad \forall j \in \mathbb{N}, \quad \text{ext}_A^i(\Delta^0(\lambda), \nabla_0(\mu)\langle j \rangle) \neq 0 \implies i = j.$$

Equivalently, using the isomorphism (1.0.1), this means that, for $0 \leq i \leq n$,

$$(2.0.8) \quad \text{Ext}_A^i(\Delta^0(\lambda), \nabla_0(\mu)) \cong \text{ext}_A^i(\Delta^0(\lambda), \nabla_0(\mu)\langle i \rangle).$$

When $A_0 = k$, the notion of an n -Q-Koszul algebra identifies with the notion of an n -Koszul algebra defined in [39, p. 29].

A graded algebra A is called Q-Koszul provided that it is n -Q-Koszul for all integers $n \in \mathbb{N}$. In other words, condition (1.0.2) holds. The notion of a Q-Koszul algebra is left-right symmetric as is the notion of standard Q-Koszul introduced in §3. We generally prefer to work with left modules.

Theorem 2.1. (a) Assume that A is n -Q-Koszul for some fixed integer $n \geq 1$. For A_0 -modules X, Y , the map (2.0.6) for $r \leq n$ is an isomorphism

$$i_* : \text{Ext}_{A_0}^r(X, Y) \xrightarrow{\sim} \text{ext}_A^r(X, Y).$$

(b) Now assume that A is Q-Koszul. Then the natural functor $i_* : A_0\text{-mod} \rightarrow A\text{-grmod}$ in (2.0.5) induced by the quotient map $A \rightarrow A/A_{\geq 1} \cong A_0$ of graded algebras induces a full embedding

$$i_* : D^b(A_0\text{-mod}) \rightarrow D^b(A\text{-grmod})$$

of derived categories.

Proof. Statement (b) follows from a well-known argument, once (a) is established. To prove (a), assume that A is n -Koszul. The map (2.0.6) is an isomorphism trivially if $n = 0, 1$ as noted after (2.0.6). So assume $n > 1$ and proceed by induction on n . Let $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ be an exact sequence in $A_0\text{-mod}$ where P is A_0 -projective. Let I be an A_0 -module having a ∇^0 -filtration. Then $\text{ext}_A^m(P, I) = 0 = \text{ext}_{A_0}^m(P, I)$ for $m = n - 1, n$, using the n -Q-Koszul property, since P has a Δ^0 -filtration. By the long exact sequence of

cohomology, there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}_{A_0}^{n-1}(K, I) & \xrightarrow{\sim} & \mathrm{Ext}_{A_0}^n(M, I) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{ext}_A^{n-1}(K, I) & \xrightarrow{\sim} & \mathrm{ext}_A^n(M, I) & \longrightarrow & 0 \end{array}$$

in which the two rows are necessarily isomorphisms. By induction, the left hand vertical map is an isomorphism so the right hand vertical map $\mathrm{Ext}_{A_0}^n(M, I) \rightarrow \mathrm{ext}_A^n(M, I)$ is an isomorphism.

This proves (a) in case $N = I$ has a ∇_0 -filtration. So now assume that N is arbitrary, and form an exact sequence $0 \rightarrow N \rightarrow I \rightarrow C \rightarrow 0$ in $A_0\text{-mod}$, where I is A_0 -injective. Thus, I has a ∇_0 -filtration, so we again get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}_{A_0}^{n-1}(M, C) & \xrightarrow{\sim} & \mathrm{Ext}_{A_0}^n(M, N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{ext}_A^{n-1}(M, C) & \xrightarrow{\sim} & \mathrm{ext}_A^n(M, N) & \longrightarrow & 0 \end{array}$$

in which the horizontal maps are isomorphisms. The left hand vertical map is an isomorphism by induction, so the right hand vertical map is also an isomorphism as required. \square

The next two results consider the special cases in which A is 1- or 2-Q-Koszul. The overall outline and some of the proof are influenced by the work of Beilinson-Ginzburg-Soergel [3, §2.3] in the Koszul case, though our situation is more involved. A positively graded algebra A is, by definition, *tight* if it is generated by A_0 and A_1 . Observe this implies (and is equivalent to) the statement $A_n = \underbrace{A_1 \cdots A_1}_n$, for all $n \geq 1$. Also, $A_m \cdot A_n = A_{m+n}$ for all $m, n \in \mathbb{N}$.

Theorem 2.2. *Assume that A is 1-Q-Koszul as above, that is, A_0 is quasi-hereditary, and for $\lambda, \mu \in \Lambda$, and for all integers m ,*

$$\mathrm{ext}_A^1(\Delta^0(\lambda), \nabla_0(\mu)\langle m \rangle) \neq 0 \implies m = 1.$$

Then the graded algebra A is tight.

Proof. The exact sequence $0 \rightarrow A_{\geq 1} \rightarrow A \rightarrow A_0 \rightarrow 0$ of graded A -modules gives an exact sequence

$$\begin{aligned} \mathrm{hom}_A(A, \nabla_0(\mu)\langle m \rangle) &\xrightarrow{\alpha} \mathrm{hom}_A(A_{\geq 1}, \nabla_0(\mu)\langle m \rangle) \\ &\xrightarrow{\beta} \mathrm{ext}_A^1(A_0, \nabla_0(\mu)\langle m \rangle) \xrightarrow{\gamma} \mathrm{ext}_A^1(A, \nabla_0(\mu)\langle m \rangle) = 0 \end{aligned}$$

for all integers $m \geq 0$. The map α is necessarily 0 for all m : consider first the case $m = 0$ (where $\mathrm{hom}_A(A_{\geq 1}, \nabla_0(\mu)) = 0$), and then $m \geq 1$ (where $\mathrm{hom}_A(A, \nabla_0(\mu)\langle m \rangle) = 0$). Hence, β is an isomorphism for all $m \geq 0$. Since A is 1-Q-Koszul, it follows that $\mathrm{hom}_A(A_{\geq 1}, \nabla_0(\mu)\langle m \rangle) = 0$ if $m > 1$. (Observe that A_0 has a Δ^0 -filtration.)

Let T be the (graded) left ideal of A generated by A_1 . To show that A is generated by A_0, A_1 , it suffices to prove that $T = A_{\geq 1}$. If not, then for some $m > 1$ and μ , $\text{hom}_A(A_{\geq 1}/T, \nabla_0(\mu)\langle m \rangle) \neq 0$. Hence, $\text{hom}_A(A_{\geq 1}, \nabla_0(\mu)\langle m \rangle) \neq 0$, a contradiction. \square

Let A be a positively graded algebra and let

$$T_{A_0}(A_1) := \bigoplus_{n \geq 0} \underbrace{A_1 \otimes_{A_0} \cdots \otimes_{A_0} A_1}_n$$

be the tensor algebra of the (A_0, A_0) -bimodule A_1 (with the term for $n = 0$ set to be A_0). Generalizing the usual definition, the graded algebra A is defined to be quadratic if the multiplication map $m : T_{A_0}(A_1) \rightarrow A$, defined by $a_1 \otimes \cdots \otimes a_n \mapsto a_1 \cdots a_n$, is surjective, and if the relation ideal $I := \ker m$ is generated by its grade 2 component. That is, I is generated by the kernel W_2 of the multiplication map $A_1 \otimes_{A_0} A_1 \rightarrow A_2$. Since A_1 is an (A_0, A_0) -bimodule, it is a left module for both algebras A_0 and A_0^{op} (and, of course, the two actions commute). For $\lambda \in \Lambda$, the corresponding standard module $\Delta^{0, \text{op}}(\mu)$ for $A^{0, \text{op}}$ is defined to be a linear dual $\nabla(\lambda)^*$, viewed as a left A_0^{op} -module. It has irreducible head $L^{\text{op}}(\lambda) = L(\lambda)^*$.

Theorem 2.3. *Assume that A is 2-Q-Koszul. (Thus, A is also 1-Q-Koszul.) Then the following statements hold.*

- (a) *A is a quadratic algebra.*
- (b) *The subspace W_2 of $A_1 \otimes_{A_0} A_1$ defined above generates the kernel of the multiplication map $A \otimes_{A_0} A_1 \rightarrow A_{\geq 1}$ (respectively, $A_1 \otimes_{A_0} A \rightarrow A_{\geq 1}$) as a left A -module (respectively, as a left A^{op} -module).*
- (c) *The left A_0 -module A_1 has a Δ^0 -filtration. Also, the left A_0^{op} -module A_1 has a $\Delta^{0, \text{op}}$ -filtration.*

Proof. We first prove (c). The long exact sequence of $\text{ext}_A^\bullet(-, \nabla_0(\mu)\langle r \rangle)$ for the exact sequence $0 \rightarrow A_{>1} \rightarrow A \rightarrow A_0 \rightarrow 0$ in $A\text{-grmod}$ gives

$$(2.0.9) \quad \text{ext}_A^1(A_{\geq 1}, \nabla_0(\mu)\langle 1 \rangle) \cong \text{ext}_A^2(A_0, \nabla_0(\mu)\langle 1 \rangle) = 0.$$

(The term on the right is 0 since, by hypothesis, A is 2-Q-Koszul.)

Next, again using the long exact sequence of $\text{ext}_A^\bullet(-, \nabla_0(\mu)\langle r \rangle)$ for the exact sequence $0 \rightarrow A_{>1} \rightarrow A_{\geq 1} \rightarrow A_1 \rightarrow 0$ gives an exact sequence

$$\begin{aligned} \text{hom}_A(A_{>1}, \nabla_0(\mu)\langle 1 \rangle) &\rightarrow \text{ext}_A^1(A_1, \nabla_0(\mu)\langle 1 \rangle) \\ &\rightarrow \text{ext}_A^1(A_{\geq 1}, \nabla_0(\mu)\langle 1 \rangle) \rightarrow \text{ext}_A^1(A_{>1}, \nabla_0(\mu)\langle 1 \rangle). \end{aligned}$$

Obviously, the left hand end of the above exact sequence vanishes, while, from (2.0.4), the right hand end is also 0. Thus, using (2.0.9),

$$0 = \text{ext}_A^1(A_{\geq 1}, \nabla_0(\mu)\langle 1 \rangle) \cong \text{ext}_A^1(A_1, \nabla_0(\mu)\langle 1 \rangle).$$

Theorem 2.1(a), applied to $\text{ext}_A^1(A_1\langle -1 \rangle, \nabla_0(\mu)) = \text{ext}_A^1(A_1, \nabla_0(\mu)\langle 1 \rangle) = 0$, gives (using (1.0.1))

$$\text{Ext}_{A_0}^1(A_1, \nabla_0(\mu)) = 0, \quad \forall \mu \in \Lambda.$$

This means that A_1 has a Δ^0 -filtration, as required by (c).

Next, we prove (a) and (b). By Theorem 2.2, there is an exact sequence

$$(2.0.10) \quad 0 \rightarrow W \rightarrow A \otimes_{A_0} A_1 \xrightarrow{\phi} A_{\geq 1} \rightarrow 0$$

of graded left A -modules. Since ϕ is an isomorphism in grade 1, the graded A -module W is concentrated in grades ≥ 2 . The following claim is needed for the proof of (a); applied together with its analog for A^{op} , it also gives (b) immediately.

Claim: W is generated in grade 2 as a left A -module, i. e., $W = AW_2$.

Before proving the Claim, we show that it implies (a), that is, that A is quadratic. By construction, $W_2 \subseteq A_1 \otimes A_1 \subseteq T := T_{A_0}(A_1) = \bigoplus_{s \geq 0} A_1^{\otimes s}$, the tensor algebra of A_1 over A_0 . (Thus, $\otimes := \otimes_{A_0}$ in this proof.) Here $A_1^{\otimes 0} := A_0$. Let I be the kernel of the evident algebra surjection $\alpha = \bigoplus_s \alpha_s : T \twoheadrightarrow A$. It suffices to prove that $I = \langle W_2 \rangle$ (the ideal in T generated by $W_2 \subseteq T$). Obviously, $I_0 = 0 = I_1$ and $I_2 = W_2$. Let $s > 2$ be an integer, and assume by induction that $I_{s-1} = \langle W_2 \rangle_{s-1}$. Thus,

$$(2.0.11) \quad \langle W_2 \rangle_{s-1} A_1 = I_{s-1} A_1 = \ker \left(A_1^{\otimes s-1} \otimes A_1 = T_{s-1} \otimes A_1 \xrightarrow{\alpha_{s-1} \otimes A_1} A_{s-1} \otimes A_1 \right),$$

where the two left hand products are taken in T .¹ We need to show that $I_s = \langle W_2 \rangle_s$, or, equivalently, $I_s \subseteq \langle W_2 \rangle_s$. Consider the commutative diagram

$$(2.0.12) \quad \begin{array}{ccccc} T_s & \xleftarrow{\sim} & T_{s-1} \otimes A_1 & \xleftarrow{\sim} & T_{s-2} \otimes A_1 \otimes A_1 \\ \downarrow \alpha_s & & \downarrow \alpha_{s-1} \otimes A_1 & & \downarrow \beta_s \\ A_s & \xleftarrow{\quad} & A_{s-1} \otimes A_1 & \xlongequal{\quad} & A_{s-2}(A_1 \otimes A_1) \end{array}$$

where each map in the top row is induced by multiplication in T .² Also, β_s is the composite of the surjective map $\alpha_{s-2} \otimes A_1 \otimes A_1$ with module multiplication of A_{s-2} on $A_1 \otimes A_1 \subseteq A \otimes A_1$. Let $x \in I_s \subseteq T_s$, and let x' be the corresponding element in $A_1^{\otimes(s-1)} \otimes A_1 = T_{s-1} \otimes A_1$. Then $(\alpha_{s-1} \otimes A_1)(x')$ maps to 0 under the multiplication map $A_{s-1} \otimes A_1 \rightarrow A$, since $x \in I_s$. Thus, $(\alpha_{s-1} \otimes A_1)(x') \in W_s$, which equals $A_{s-2}W_2$ by the Claim. Since α_{s-2} is surjective, there is an element $y^\# \in A_1^{\otimes(s-2)}W_2 = T_{s-2}W_2 \subseteq T_{s-2} \otimes A_1 \otimes A_1$ with image $(\alpha_{s-1} \otimes A_1)(x') \in W_s = A_{s-2}W_2$ under the map β_s . Let $y' \in T_{s-1} \otimes A_1$ and $y \in T_s$ correspond to $y^\#$ in the top row of (2.0.12). The commutativity of (2.0.12) gives $(\alpha_{s-1} \otimes A_1)(y') = (\alpha_{s-1} \otimes A_1)(x')$. Then $x' - y' \in \ker(\alpha_{s-1} \otimes A_1)$, which by induction is $\langle W_2 \rangle_{s-1} A_1$, so $x - y \in \langle W_2 \rangle_{s-1} A_1 \subseteq T_s$ in (2.0.11). By construction, y' and y belong to

¹Of course, $I_{s-1} \subseteq A_1^{\otimes(s-1)}$, though there may not be an inclusion of $I_{s-1} \otimes A_1$ into $A_1^{\otimes(s-1)} \otimes A_1$ because the functor $- \otimes A_1$ might possibly be only right exact. Nevertheless, it makes sense to form the product $I_{s-1} A_1$ in T . This product is the same as the image of the map $I_{s-1} \otimes A_1 \rightarrow A_1^{\otimes(s-1)} \otimes A_1 = T_s$. The right exactness of $- \otimes A_1$ then gives (2.0.11).

²In some sense, these multiplication maps are just equalities, but it is useful in the proof to keep them separate.

$\langle W_2 \rangle$. So $x \in \langle W_2 \rangle$. This completes the inductive step $I_s = \langle W_2 \rangle_s$. Thus, $I = \langle W_2 \rangle$, and A is quadratic.

It remains to check the Claim (which will also prove (b)). First, there is an injection

$$\mathrm{hom}_A(W, \nabla_0(\lambda)\langle r \rangle) \hookrightarrow \mathrm{ext}_A^2(A_0, \nabla_0(\lambda)\langle r \rangle), \quad \forall r \in \mathbb{N}.$$

To see this, let P be an A_0 -projective cover of A_1 , viewed as graded and concentrated in grade 1. Then $A \otimes P = A \otimes_{A_0} P$ is a graded projective A -module, equipped with a map $A \otimes P \rightarrow A_{\geq 1}$ (surjective, by Theorem 2.2) sending $1 \otimes P$ to A_1 . Let \widehat{W} be the kernel of this map. There is a commutative diagram

$$(2.0.13) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{W} & \longrightarrow & A \otimes P & \longrightarrow & A & \longrightarrow & A_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & W & \longrightarrow & A \otimes A_1 & \longrightarrow & A & \longrightarrow & A_0 & \longrightarrow & 0, \end{array}$$

in which each row is exact and the vertical maps are the evident surjections. The top row can be used to compute $\mathrm{ext}_A^2(A_0, \nabla_0(\lambda)\langle r \rangle)$ in the usual cocycles/coboundaries way. The 2-cocycles are elements of $\mathrm{hom}_A(\widehat{W}, \nabla_0(\lambda)\langle r \rangle)$, and the image in it of $\mathrm{hom}_A(A \otimes P, \nabla_0(\lambda)\langle r \rangle)$ is the space of 2-coboundaries. All elements of $\mathrm{hom}_A(A \otimes P, \nabla_0(\lambda)\langle r \rangle)$ are zero, unless $r = 1$, since P is concentrated in grade 1. But $\mathrm{hom}_A(W, \nabla_0(\lambda)\langle r \rangle) = 0$ when $r = 1$ (since $W_m = 0$ for $m \leq 1$). The composite of the maps

$$(2.0.14) \quad \mathrm{hom}_A(W, \nabla_0(\lambda)\langle r \rangle) \rightarrow \mathrm{hom}_A(\widehat{W}, \nabla_0(\lambda)\langle r \rangle) \rightarrow \mathrm{ext}_A^2(A_0, \nabla_0(\lambda)\langle r \rangle)$$

is an injection in all cases.

Observe that W/AW_2 is a (positively) graded A -modules, vanishing in grades ≤ 2 . If $W/AW_2 \neq 0$, we have $(W/AW_2)_s \neq 0$ for some minimal integer s . Necessarily $s > 2$. Also, $(W/AW_2)_s = (AW_2 + W_{\geq s})/(AW_2 + W_{> s})$ is an A -module killed by A_1 . Choose any irreducible graded $A_0 = A/A_{\geq 1}$ -module $L(\lambda)\langle s \rangle$ in the head of $(W/AW_2)_s$. Then $\mathrm{hom}_A(W, \nabla_0(\lambda)\langle s \rangle) \neq 0$, so $\mathrm{ext}_A^2(A_0, \nabla_0(\lambda)\langle s \rangle) \neq 0$ by (2.0.14). Since $s > 2$, this contradicts the hypothesis is 2-Q-Koszul. This contradiction shows that $W = AW_2$ and completes the proof of the Claim. Thus, (a) and (b) are now proved, as well as the theorem. \square

If V is an A_0 -module, let $\cdots \rightarrow P^1 \rightarrow P^0 \twoheadrightarrow V$ be a minimal projective resolution in the category of A_0 -modules. Recall, for any $m > 0$, the m th syzygy module $\Omega_m(V)$, for the algebra A_0 , is the kernel of the map $P^{m-1} \rightarrow P^{m-2}$ (setting $P^{-1} := V$). By convention, $\Omega_0(V) = V$.

Lemma 2.4. *Assume that A is $(n+1)$ -Q-Koszul for some integer $n > 0$. Let s, r be non-negative integers with $s \leq r$ and $0 < s \leq n$.*

(a) *Then $\mathrm{Ext}_{A_0}^r(A_s, \nabla_0(\gamma)) = 0$ for all $\gamma \in \Lambda$. In particular, the A_0 -modules*

$$\Omega_{r-1}(A_s), \Omega_r(A_s), \Omega_{r+1}(A_s), \dots$$

all have Δ^0 -filtrations. Consequently (using the case $r = s$), the A_0 -modules

$$\Omega_{s-1}(A_s), \Omega_s(A_s), \Omega_{s+1}(A_s), \dots$$

all have Δ^0 -filtrations.

(b) Assume also $r \leq n + 1$, and let m be any integer such $m \leq r$. Then

$$(2.0.15) \quad \text{ext}_A^r(A_s, \nabla_0(\nu)\langle m \rangle) = 0, \quad \forall \nu \in \Lambda.$$

Proof. Before beginning the proof, we observe an additional consequence (c) of the hypotheses of the lemma. It is obvious, since A_0 has a Δ^0 -filtration (as a left A_0 -module) by standard properties of QHAs.

(c) Let r be a nonnegative integer $\leq n + 1$. Then (2.0.15) holds for $s = 0$ and all $m \neq r$.

In order to prove (a) for a given $s > 0$, it is sufficient to prove the shorter statement:

$$(a') \quad \text{Ext}_{A_0}^s(A_s, \nabla_0(\nu)) = 0, \quad \forall \nu \in \Lambda.$$

In fact, assume that (a') holds for s . Dimension shifting gives $\text{Ext}_{A_0}^1(\Omega_{s-1}(A_s), \nabla_0(\nu)) = 0$, for all $\nu \in \Lambda$. Thus, $\Omega_{s-1}(A_s)$ has a Δ^0 -filtration. This also implies the higher syzygies $\Omega_s(A_s), \Omega_{s+1}(A_s), \dots$ have Δ^0 -filtrations, or, equivalently, $\text{Ext}_{A_0}^r(A_s, \nabla_0(\nu)) = 0$ for all $\nu \in \Lambda$ and $r \geq s$. This completes the proof that (a') \implies (a), for any given $s > 0$. The opposite implication, which is not used below, is obvious.

We prove part (a) by induction on n . The main isomorphism we develop will also help to prove part (b). Thus, if $0 < s < n$, (a) holds as written for s by induction. In particular, the A_0 -syzygy $\Omega_{s-1}(A_s)$ has a Δ^0 -filtration. Once part (a) has been proved for n , we can also allow $s = n$ in this statement, i. e., we will be able to conclude that $\Omega_{n-1}(A_n)$ has a Δ^0 -filtration.

Using the map $\pi : A \rightarrow A_0$, regard $X := \Omega_{s-1}(A_s)$ as an object in A -grmod which is pure of grade 0. Thus, X is the end term in an evident partial resolution of $A_s\langle -s \rangle$, the latter viewed as a purely graded A -module $(A_{\geq s}/A_{> s})\langle -s \rangle$ of grade 0. The intermediate terms in this partial resolution are projective A_0 -modules P , viewed as purely graded A -modules of grade 0. With the latter interpretation of P , we have, for any integer m , $\text{ext}_A^r(P, \nabla_0(\nu)\langle m - s \rangle) = 0$ whenever $n + 1 \geq r > m - s$, since P has a Δ^0 -filtration, and we have assumed the $(n + 1)$ -Q-Koszul property. This vanishing may be used to iteratively dimension shift, starting with $r \leq n + 1$ and ending with $r - (s - 1)$, provided $r - (s - 1) > m - s$ and $r - (s - 1) > 0$. Equivalently, $r \geq m$ and $r \geq s$. This gives the lower isomorphism below, for $m \leq r \leq n + 1$, $s \leq r$, and $0 < s < n$,

$$(2.0.16) \quad \begin{aligned} \text{ext}_A^r(A_s, \nabla_0(\nu)\langle m \rangle) &\cong \text{ext}_A^r(A_s\langle -s \rangle, \nabla_0(\nu)\langle m - s \rangle) \\ &\cong \text{ext}_A^{r-s+1}(X, \nabla_0(\nu)\langle m - s \rangle). \end{aligned}$$

The lower term is 0 in all these cases, since X has a Δ^0 -filtration, and $r - s + 1 \neq m - s$. In particular, these isomorphisms and vanishings hold for $r = n + 1$ and $m = n$, for any positive integer $s < n$. Also, with the (same) values $r = n + 1$ and $m = n$, we have the additional vanishing $\text{ext}_A^r(A_s, \nabla_0(\nu)\langle m \rangle) = 0$ when $s = 0$, by (c) above. Consequently,

noting that the graded quotient $A_{<n}$ of the graded A -module A is filtered by the graded A -modules A_s , $0 \leq s < n$, we have (as in the proof of Theorem 2.3(c))

$$\mathrm{ext}_A^n(A_n, \nabla_0(\nu)\langle n \rangle) \cong \mathrm{ext}_A^n(A_{\geq n}, \nabla_0(\nu)\langle n \rangle) \cong \mathrm{ext}_A^{n+1}(A_{<n}, \nabla_0(\nu)\langle n \rangle) = 0.$$

Hence, $\mathrm{Ext}_{A_0}^n(A_n, \nabla_0(\nu)) = 0$ by Theorem 2.1. This proves (a') and, thus, statement (a) for $s = n$, completing the inductive step. This proves (a).

In particular, we can now use the case $s = n$ in the above discussions. The displayed isomorphisms (2.0.16), now allowing $s = n$ as well, give all the vanishings required by part (b). This completes the proof. \square

Let A is a QHA with weight poset Λ , and let Γ be a non-empty ideal in Λ . For any nonempty poset ideal Γ in Λ , let A_Γ be the largest quotient algebra of A whose modules consist of all finite dimensional A -modules with composition factors $L(\gamma)$, $\gamma \in \Gamma$.

Lemma 2.5. *Let m be a non-negative integer. Assume A is an algebra that is both m -Q-Koszul and quasi-hereditary with weight poset Λ . Let Γ be a non-empty poset ideal in Λ . Then A_Γ is also m -Q-Koszul and quasi-hereditary, with weight poset Γ .*

Proof. First, $A_\Gamma = A/J$ for some idempotent ideal J in A . Then [7] implies that $J = AeA$ for some $e \in A_0$. It follows that A_Γ is positively graded, and the natural map $A \rightarrow A_\Gamma$ is a homomorphisms of graded algebras. If $\gamma \in \Gamma$, $\Delta^0(\gamma)$ is the standard object corresponding for $(A_\Gamma)_0$ to the weight γ . Similarly, $\nabla_0(\gamma)$ is the costandard object for $(A_\Gamma)_0$. Now the result follows from the naturally of (1.0.1), together with standard recollement properties of QHAs. \square

Remark 2.6. Although it is not used in this paper, it can be easily shown that, in the notation of the above proof, the algebra eAe is m -Q-Koszul and quasi-hereditary. The module category $eAe\text{-mod}$ is equivalent to the quotient category of $A\text{-mod}$ by the subcategory which is strict image of $A/J\text{-mod}$ in $A\text{-mod}$. For a similar result, in the setting of standard Q-Koszul algebras, see the ‘‘recollement’’ discussion at the end of §3.

If A is a positively graded QHA with weight poset Λ , then each standard module $\Delta(\gamma)$ can be graded $\Delta(\gamma) = \bigoplus_{n \geq 0} \Delta(\gamma)_n$, with $\Delta(\gamma)_0 \cong \Delta^0(\gamma)$, the standard object for the QHA A_0 . We have the following result.

Corollary 2.7. *Let A be 2-Q-Koszul and quasi-hereditary with weight poset Λ .*

(a) *For $\gamma \in \Lambda$, $\Delta(\gamma)_1$ has a Δ^0 -filtration.*

(b) *If A is 3-Q-Koszul, then $\mathrm{Ext}_{A_0}^r(\Delta(\gamma)_2, \nabla_0(\nu)) = 0$ for all $r \geq 2$ and all $\nu \in \Lambda$.*

More generally, assume that A is $(n+1)$ -Q-Koszul for some integer $n > 0$, in addition to being a QHA, and let $s \leq r$ be nonnegative integers with $0 < s \leq n$. Then $\mathrm{Ext}_{A_0}^r(\Delta(\gamma)_s, \nabla_0(\nu)) = 0$ for all $\gamma, \nu \in \Lambda$.

In addition, the syzygy modules $\Omega_{s-1}(\Delta(\gamma)_s)$, $\Omega_s(\Delta(\gamma)_s)$, $\Omega_{s+1}(\Delta(\gamma)_s)$, \dots all have Δ^0 -filtrations.

Proof. We first prove (a). If $\gamma \in \Lambda$ is maximal, then $\Delta(\gamma)$ is a projective graded A -module (with $\Delta(\gamma)_0$ identifying with the A_0 -head of $\Delta(\gamma)$). Then, by Theorem 2.3(c), it follows

that $\Delta(\gamma)_1$ has a Δ^0 -filtration. If γ is not maximal, we can choose an ideal Γ which contains γ as a maximal element. Part (a) reduces to the case in which A is replaced by A_Γ , using Lemma 2.5, and (a) follows as above.

Part (b) and the remaining paragraph are proved similarly. \square

We can now prove the following result. We assume that A is a positively graded QHA which is 2-Q-Koszul. As in Theorem 2.3, let W_2 be the kernel of the multiplication map $A_1 \otimes_{A_0} A_1 \rightarrow A_2$. An $A_0 \otimes_k A_0^{\text{op}}$ -module (equivalently, and (A_0, A_0^{op}) -bimodule) M has, by definition, a $\Delta^0 \otimes_k \Delta^{0,\text{op}}$ -filtration if and only if it has a submodule filtration with sections $\Delta^0(\lambda) \otimes_k \Delta^{0,\text{op}}(\mu)$, for $\lambda, \mu \in \Lambda$. For example, the algebra A_0 , viewed as an $A_0 \otimes_k A_0^{\text{op}}$ -module has a filtration with sections $\Delta^0(\lambda) \otimes_k \Delta^{0,\text{op}}(\lambda)$, $\lambda \in \Lambda$. It will be useful to keep in mind that the tensor product $A \otimes_k B$ of QHAs A and B over k is again quasi-hereditary. If Λ_A and Λ_B are the posets of A and B , then $\Lambda = \Lambda_A \times \Lambda_B$ is the poset of $A \otimes_k B$, with $(\lambda, \lambda') \leq (\mu, \mu')$ if and only if $\lambda \leq \mu$ and $\lambda' \leq \mu'$. The standard (respectively, costandard) modules for $A \otimes_k B$ are tensor products of standard (respectively, costandard) modules of A with those of B . For more details, see [44].

Theorem 2.8. *Assume that A is 2-Q-Koszul and that A is a QHA.*

(a) *Then the (A_0, A_0^{op}) -bimodule A_1 has a $\Delta^0 \otimes_k \Delta^{0,\text{op}}$ -filtration. Also, $A_1 \otimes_{A_0} A_1$ has a $\Delta^0 \otimes_k \Delta^{0,\text{op}}$ -filtration.*

(b) *Now assume, in addition, that A is 3-Q-Koszul. Then W_2 (defined above) has a $\Delta^0 \otimes_k \Delta^{0,\text{op}}$ -filtration.*

Proof. The $A \otimes_k A^{\text{op}}$ -module A has a filtration with sections $\Delta(\lambda) \otimes_k \Delta^{\text{op}}(\lambda)$, $\lambda \in \Lambda$, as briefly discussed in the proof of Lemma 2.5. Consequently, if $s \geq 0$, the $(A_0 \otimes_k A_0^{\text{op}})$ -module A_s has a filtration with sections $(\Delta(\lambda) \otimes_k \Delta^{\text{op}}(\lambda))_s$. The tensor product $\Delta(\lambda) \otimes_k \Delta^{\text{op}}(\lambda)$ arises as part of the image of a product $Ae \cdot eA$ in a quotient A/J of A by a graded idempotent ideal J . In fact, $\Delta(\lambda)$ arises as the image of Ae , and $\Delta^{\text{op}}(\lambda)$ identifies with the image of eA . Clearly, $(Ae)_i \cdot (eA)_j \subseteq (AeA)_{i+j}$. Consequently, there is an identification of $A_0 \otimes_k A_0^{\text{op}}$ -modules

$$(2.0.17) \quad (\Delta(\lambda) \otimes_k \Delta^{\text{op}}(\lambda))_s \cong \bigoplus_{i+j=s} \Delta(\lambda)_i \otimes_k \Delta^{\text{op}}(\lambda)_j.$$

For $s = 1$, it is now clear that A_1 has a $\Delta^0 \otimes_k \Delta_0^{\text{op}}$ -filtration. This proves the first statement in (a). The second assertion in (a) follows from the fact that

$$\Delta^0(\lambda) \otimes_{A_0}^{\mathbb{L}} \Delta^{0,\text{op}}(\mu) \cong \begin{cases} k, & \text{if } \lambda = \mu; \\ 0 & \text{otherwise.} \end{cases}.$$

See [35, Prop. 9.1].

To begin the proof of (b), we first apply the Künneth formula to the terms in the direct sum on the right hand side of (2.0.17), to obtain

$$(2.0.18) \quad \text{Ext}_{A_0 \otimes_k A_0^{\text{op}}}^m(\Delta(\lambda)_i \otimes_j \Delta^{\text{op}}(\lambda)_j, \nabla_0(\nu) \otimes_k \nabla_{0,\text{op}}(\mu)) \cong \bigoplus_{u+v=m} \text{Ext}_{A_0}^u(\Delta(\lambda)_i, \nabla_0(\nu)) \otimes_k \text{Ext}_{A_0^{\text{op}}}^v(\Delta^{\text{op}}(\lambda)_j, \nabla_{0,\text{op}}(\mu)).$$

We know that $\text{Ext}_{A_0}^u(\Delta(\lambda)_i, \nabla_0(\nu)) = 0$ for $0 < i \leq u$ and $i \leq 2$, by Corollary 2.7. A similar vanishing holds, of course, for $i = 0$, if $u > 0$. Also, we can work with A_0^{op} , to obtain that $\text{Ext}_{A_0^{\text{op}}}^v(\Delta^{\text{op}}(\lambda)_j, \nabla_{0,\text{op}}(\nu)) = 0$ for $0 < j \leq v$ and $j \leq 2$.

Claim: $\text{Ext}_{A_0 \otimes_k A_0^{\text{op}}}^2(A_2, \nabla(\lambda) \otimes_k \nabla_{0,\text{op}}(\nu)) = 0, \quad \forall \lambda, \nu.$

To prove this, we take $m = s = 2$ in (2.0.17) and (2.0.18). Because, as noted above, A_2 has a filtration with sections $(\Delta(\lambda) \otimes_k \Delta^{\text{op}}(\lambda))_s$, it suffices to show each term in the sum (2.0.18) is 0 when $i + j = s$, $i, j \geq 0$. This is clear if either $u = 2$ or $v = 2$ from the vanishing results immediately above the Claim. The other case is $u = v = 1$. Then, if $i = j = 1$, we are done by Corollary 2.7. Otherwise, either $i = 0$ or $j = 0$, and the discussion immediately above the Claim again applies.

Finally, to complete the proof, consider the short exact sequence $0 \rightarrow W_2 \rightarrow A_1 \otimes_{A_0} A_1 \rightarrow A_2 \rightarrow 0$ and the resulting long exact sequence of $\text{Ext}_{A_0 \otimes_k A_0^{\text{op}}}^\bullet(-, \nabla_0(\lambda) \otimes_k \nabla_{0,\text{op}}(\mu))$. Now (b) follows from the Claim and part (a). \square

Remark 2.9. In general, it may not true that $\Delta(\lambda)_2$ has a Δ^0 -filtration under the hypothesis of Theorem 2.8(b). This follows from the discussion of Turner's counterexample in section 5 below. In addition, this means that A_2 may not have a Δ^0 -filtration under these hypotheses. For graded algebras A arising from semisimple algebraic groups and finite posets of p -regular weights, all the A_0 -modules A_r have Δ^0 -filtrations if $p \gg 0$, using [33, Thm. 5.1]. (Using the argument from the proof of Theorem 2.8, it follows each A_r has, in fact, a $\Delta^0 \otimes_k \Delta^{0,\text{op}}$ -filtration, under this $p \gg 0$, p -regular weight hypothesis.) However, our aim in this section has been to develop a theory which might hold for small primes, including even $p = 2$ in type A , where Turner's counterexample occurs. See the conjectures in Section 7. The broad class of examples proposed there is expected to be at least Q-Koszul and quasi-hereditary, and even satisfy the stronger "standard Q-Koszul" property discussed in the next section.

Problem 2.10. Let M be a graded module for a Q-Koszul algebra A . Give conditions on a resolution of M equivalent to the condition that $\text{ext}^m(M, \nabla_0(\lambda)\langle r \rangle) \neq 0 \implies m = r$, for any $\lambda \in \Lambda$.

3. STANDARD Q-KOSZUL ALGEBRAS

In this section, standard Q-Koszul algebras are defined. The definition simplifies that given in [35], but a main result establishes the two different notions are the same.

Suppose that B is a QHA with weight poset Λ over an algebraically closed field. Let $\mathcal{C} = \mathcal{C}(B)$ be the (highest weight) category of finite dimensional B -modules. If Γ is a non-empty ideal in Λ , let $\mathcal{C}[\Gamma]$ be the full subcategory of B -modules which have composition factors $L(\gamma)$, $\gamma \in \Gamma$. Of course, $\mathcal{C}[\Gamma] = \mathcal{C}(B/J)$, for a suitable defining ideal $J = J(\Gamma)$ of B . Necessarily B/J is a QHA and $\mathcal{C}[\Gamma]$ is a highest weight category with weight poset Γ . For details, see [6].

If $B = \bigoplus_{n \geq 0} B_n$ is positively graded, let $\mathcal{C}_{\text{gr}} = \mathcal{C}_{\text{gr}}(B)$ be the category of finite dimensional \mathbb{Z} -graded B -modules. (Sometimes, we also denote \mathcal{C}_{gr} by $B\text{-grmod}$.) By [7, Prop. 4.2], the idempotent ideal $J = J(\Gamma)$ is homogeneous; in fact, $J = BeB$ for some idempotent $e \in B_0$. Each standard module $\Delta(\lambda)$, $\lambda \in \Lambda$, has a natural \mathbb{N} -grading $\Delta(\lambda) = \bigoplus_{i \geq 0} \Delta^i(\lambda)$ in which each $\Delta_i(\lambda)$ is naturally a B_0 -module. Similarly, the costandard module $\nabla(\lambda)$ has a grading $\nabla(\lambda) = \bigoplus_{i \leq 0} \nabla_i(\lambda)$.

A proof of the following elementary result is found in [33, Cor. 3.2].

Lemma 3.1. *Suppose $B = \bigoplus_{n \geq 0} B_n$ is a positively graded quasi-hereditary algebra with poset Λ . Then the subalgebra B_0 is quasi-hereditary with poset Λ . For $\lambda \in \Lambda$, the corresponding standard (respectively, costandard) module is $\Delta^0(\lambda)$ (respectively, $\nabla_0(\lambda)$).*

Definition 3.2. The graded quasi-hereditary algebra B is called a *standard Q-Koszul algebra* provided that, for all $\lambda, \mu \in \Lambda$,

$$(3.0.19) \quad \begin{cases} (a) & \text{ext}_B^n(\Delta(\lambda), \nabla_0(\mu)\langle r \rangle) \neq 0 \implies n = r; \\ (b) & \text{ext}_B^n(\Delta^0(\mu), \nabla(\lambda)\langle r \rangle) \neq 0 \implies n = r \end{cases}$$

for all integers n, r .

Consider the bounded derived category $D^b(\mathcal{C}_{\text{gr}})$ for the abelian category $\mathcal{C}_{\text{gr}} = \mathcal{C}_{\text{gr}}(B)$ of (finite dimensional) graded B -modules. Then $D^b(B\text{-grmod})$ is a triangulated category with shift operator $X \mapsto X[1]$, $X \in D^b(B\text{-grmod})$. If $X = X^\bullet$ is represented by a complex in $D^b(B\text{-grmod})$, $X[1] \in D^b(B\text{-grmod})$ is the complex obtained by shifting X one-unit to the left and replacing each differential by its negative. Put $[r] = [1]^r$. Next, set $X\langle r \rangle$ to be the complex by applying the grading shift operator $\langle r \rangle$ to the terms X^n and the differentials. Finally, let $X\{r\} := X\langle r \rangle[r] = X[r]\langle r \rangle$.

Define a full subcategory $\mathcal{E}^L = \mathcal{E}^L(\mathcal{C}_{\text{gr}}) := \bigcup_{i \geq 0} \mathcal{E}_i^L$ of $D^b(\mathcal{C}_{\text{gr}})$ as follows: Let $\mathcal{E}_0^L \subseteq D^b(\mathcal{C}_{\text{gr}})$ consist of all finite direct sums $\Delta(\lambda)\{r\}$, for $\lambda \in \Lambda$, $r \in \mathbb{Z}$. Having defined \mathcal{E}_r^L define \mathcal{E}_{r+1}^L to consist of all objects $X \in D^b(\mathcal{C}_{\text{gr}})$ for which there is a distinguished triangle $Y \rightarrow X \rightarrow Z \rightarrow$ with $Y, Z \in \mathcal{E}_r^L$. Another full subcategory $\mathcal{E}^R := \mathcal{E}^R(\mathcal{C}_{\text{gr}}) = \bigcup_{i \geq 0} \mathcal{E}_i^R$ of $D^b(B\text{-grmod})$ is constructed similarly, but using the $\nabla(\lambda)\{r\}$, $\lambda \in \Lambda$, $r \in \mathbb{Z}$.

For $X, Y \in D^b(B\text{-grmod})$ and $n \in \mathbb{Z}$, write $\text{hom}^n(X, Y) := \text{Hom}_{D^b(\mathcal{C}_{\text{gr}})}(X, Y[n])$. If $X, Y \in \mathcal{C}_{\text{gr}}$ are viewed as complexes concentrated in grade 0, then $\text{hom}^n(X, Y) = \text{ext}_B^n(X, Y)$ for $n \geq 0$, and $= 0$ if $n < 0$.

Theorem 3.3. *Given $M \in D^b(\mathcal{C}_{\text{gr}})$,*

$$\begin{cases} (a) & M \in \mathcal{E}^L \iff \forall \lambda \in \Lambda, n, r \in \mathbb{Z}, \text{hom}_{D^b(\mathcal{C}_{\text{gr}})}^n(M, \nabla(\lambda)\langle r \rangle) \neq 0 \implies n = r \\ (b) & M \in \mathcal{E}^R \iff \forall \lambda \in \Lambda, n, r \in \mathbb{Z}, \text{hom}_{D^b(\mathcal{C}_{\text{gr}})}^n(\Delta(\lambda)\langle r \rangle, M) \neq 0 \implies n = -r. \end{cases}$$

Proof. We will prove statement (a), leaving the similar (b) to the reader. For $M \in D^b(\mathcal{C}_{\text{gr}})$, we say that condition $(\star(M))$ holds provided:

$$\forall \lambda \in \Lambda, n, r \in \mathbb{Z}, \text{hom}_{D^b(\mathcal{C}_{\text{gr}})}^n(M, \nabla(\lambda)\langle r \rangle) \neq 0 \implies n = r.$$

Also, we can assume (after a refinement consistent with the original partial ordering) that the poset Λ is totally ordered.

(\Rightarrow) First,

$$\text{hom}_{D^b(\mathcal{C}_{\text{gr}})}^n(\Delta(\lambda)\{a\}, \nabla(\mu)\langle b \rangle) \cong \text{hom}_{D^b(\mathcal{C}_{\text{gr}})}^{n-a}(\Delta(\lambda), \nabla(\mu)\langle b-a \rangle).$$

Hence, if the left hand side is non-zero, necessarily $\text{Ext}_B^{n-a}(\Delta(\lambda), \nabla(\mu)) \neq 0$ so $n-a=0$ and $\lambda = \mu$ by well-known homological properties of standard and co-standard modules. Hence, $\text{hom}_{\mathcal{C}_{\text{gr}}}(\Delta(\lambda), \nabla(\lambda)\langle b-a \rangle) \neq 0$ so $b-a=0$. Thus, $n-a=b-a$ so $n=b$, as required. Now it follows that $(\star(M))$ holds if M is a direct sum of objects $\Delta(\lambda)\{m\}$, $m \in \mathbb{Z}$. Finally, if $N \rightarrow M \rightarrow Q \rightarrow$ is a distinguished triangle in $D^b(\mathcal{C}_{\text{gr}})$ and, if both $(\star(N))$ and $(\star(Q))$ hold, then $(\star(M))$ holds since hom is a cohomological bifunctor (i. e., takes distinguished triangles in either variable to long exact sequences). Thus, $(\star(M))$ holds on \mathcal{E}^L , as required.

(\Leftarrow) Consider the set Ξ of ordered pairs $(|\Gamma|, m)$, where Γ is an ideal (possibly the empty ideal) in Λ , and m is a positive integer. The set Ξ is ordered lexicographically. Given $X \in D^b(\mathcal{C}_{\text{gr}})$, let $d(X) := (|\Gamma|, m)$, where Γ is the ideal generated by the maximal element $\gamma \in \Lambda$ for which $[H^\bullet(X) : L(\gamma)] \neq 0$ and $m = [H^\bullet(X) : L(\gamma)]$. If $X = 0$, $d(X) = (0, 0)$.

Assume that $M \in D^b(\mathcal{C}_{\text{gr}})$ satisfies the condition $(\star(M))$. We must show that $M \in \mathcal{E}^L$. We proceed by induction on $d(M)$ for M satisfying $(\star(M))$. If $d(M) = (0, 0)$, then $M \cong 0$ and so $M \in \mathcal{E}^L$, trivially. So assume that $d(M) \neq (0, 0)$. Let $d(M) = (|\Gamma|, m)$, and observe that because Λ is totally ordered, the cardinality of Γ determines Γ . Let $\gamma \in \Gamma$ be the unique maximal element. Since $H^\bullet(M) \in \mathcal{C}_{\text{gr}}[\Gamma]$, if $J = J(\Gamma)$, then M belongs to the relative derived category $D_{\mathcal{C}_{\text{gr}}(B/J)}^b(\mathcal{C}_{\text{gr}})$ which can be identified with $D^b(\mathcal{C}_{\text{gr}}[\Gamma])$ —see [6, §2.3] and the references there. It suffices to show that $M \in \mathcal{E}^L(B/J)$, since the natural full embedding $i_* : D^b(\mathcal{C}_{\text{gr}}(B/J)) \rightarrow D^b(\mathcal{C}_{\text{gr}}(B))$ induced by the quotient map $B \twoheadrightarrow B/J$ carries $\mathcal{E}^L(B/J)$ to $\mathcal{E}^L(B)$, by the inductive definition of $\mathcal{E}^L(B/J)$.

For some choice of integers t and r , $L(\gamma)\langle t \rangle$ is a composition factor of $H^r(M)$ in $\mathcal{C}_{\text{gr}}(B/J)$. Because $\nabla(\gamma) \in B/J\text{-grmod}$ is an injective module,

$$0 \neq \text{hom}_{\mathcal{C}_{\text{gr}}}(H^r(M), \nabla(\gamma)\langle t \rangle) \cong \text{hom}_{D^b(\mathcal{C}_{\text{gr}})}^{-r}(M, \nabla(\gamma)\langle t \rangle).$$

In particular, $t = -r$ since $(\star(M))$ holds. Choose a morphism $f : \Delta(\gamma)\{t\} \rightarrow M$ inducing a surjection

$$\text{hom}_{D^b(\mathcal{C}_{\text{gr}})}^t(M, \nabla(\gamma)\langle t \rangle) \twoheadrightarrow \text{hom}_{D^b(\mathcal{C}_{\text{gr}})}^t(\Delta(\gamma)\{t\}, \nabla(\gamma)\langle t \rangle) \cong k.$$

Consequently, for each integer n , there is a surjection

$$\mathrm{hom}_{D^b(\mathcal{C}_{\mathrm{gr}})}^n(M, \nabla(\gamma)\langle t \rangle) \twoheadrightarrow \mathrm{hom}_{D^b(\mathcal{C}_{\mathrm{gr}})}^n(\Delta(\gamma)\{t\}, \nabla(\gamma)\langle t \rangle)$$

(for $n \neq t$ both sides are 0). Now form the distinguished triangle $\Delta(\gamma)\{r\} \rightarrow M \rightarrow M' \rightarrow$ and observe that

$$\mathrm{hom}^n(M', \nabla(\lambda)\langle t \rangle) \subseteq \mathrm{hom}^n(M, \nabla(\lambda)\langle t \rangle), \quad \forall n, \lambda$$

since hom is a cohomological bifunctor. In particular, this means that $(\star(M'))$ holds.

Since $M' \in D^b(\mathcal{C}_{\mathrm{gr}}[\Gamma])$ the composition factors $L(\gamma')$ of $H^\bullet(M')$ all satisfy $\gamma' \in \Gamma$. However, $[H^n(M') : L(\gamma)] = [H^n(M) : L(\gamma)]$ if $n \neq r$, while $[H^r(M') : L(\gamma)] < [H^r(M) : L(\gamma)]$. Thus, $d(M') < d(M)$. By induction, $M' \in \mathcal{E}^L$. Since $\Delta(\gamma)\{r\} \in \mathcal{E}^L$ as well, and $\Delta(\gamma)\{t\} \rightarrow M \rightarrow M' \rightarrow$ is distinguished, it finally follows that $M \in \mathcal{E}^L$. \square

Corollary 3.4. *Assume that B is a standard Q -Koszul algebra with weight poset Λ . For $\lambda, \mu \in \Lambda$,*

$$\mathrm{ext}_B^n(\Delta^0(\lambda), \nabla_0(\mu)\langle r \rangle) \neq 0 \implies n = r.$$

Therefore, B is a Q -Koszul algebra.

Proof. By Theorem 3.3, $\Delta^0(\lambda) \in \mathcal{E}^L$ and $\nabla_0(\mu) \in \mathcal{E}^R$. Using the definition of \mathcal{E}^L , it is enough to check that

$$\mathrm{hom}^n(\Delta(\rho)\{a\}, \nabla_0(\mu)\langle r \rangle) \neq 0 \implies n = r.$$

But

$$\mathrm{hom}^n(\Delta(\rho)\{a\}, \nabla_0(\mu)\langle r \rangle) \cong \mathrm{hom}^{n-a}(\Delta(\rho), \nabla_0(\mu)\langle r - a \rangle),$$

so that $n - a = r - a$ or $n = r$ as required. A similar argument applies to \mathcal{E}^R . \square

We also have the following consequence of the above corollary together with Theorems 2.3 and 2.8.

Corollary 3.5. *Assume that B is a standard Q -Koszul algebra with weight poset Λ . Then B is a quadratic algebra. In addition, the $B \otimes_k B^{\mathrm{op}}$ -modules B_1 and $B_1 \otimes_{B_0} B_1$ each have a $\Delta^0 \otimes_k \Delta^{0, \mathrm{op}}$ -filtration, as does the grade 2 relation module W_2 (defined above Theorem 2.3).*

Recollement

Finally, we indicate how recollement works for standard Q -Koszul algebras. More specifically, let B be standard Q -Koszul with poset Λ . Let Γ be a non-empty ideal in Λ . Let $B\text{-mod}[\Gamma]$ be the full subcategory of $B\text{-mod}$ consisting of all finite dimensional modules having composition factors $L(\gamma)$, $\gamma \in \Gamma$. Then $B\text{-mod}[\Gamma] \cong B/J\text{-mod}$, for some idempotent ideal J in B . By [11], $J = BeB$ for some idempotent $e \in B_0$, so that B/J is a positively graded quasi-hereditary algebra, and we can form the category $B/J\text{-grmod}$ of graded B/J -modules. As remarked before, the quotient map $B \twoheadrightarrow B/J$ defines full embeddings $i_* : D^b(B/J\text{-mod}) \rightarrow D^b(B\text{-mod})$ and $i_* : D^b(B/J\text{-grmod}) \rightarrow D^b(B\text{-grmod})$. The functor i_* takes standard modules $\Delta^{B/J}(\gamma)$ for the quasi-hereditary algebra B/J to

standard modules $\Delta^B(\gamma) = \Delta(\gamma)$, and we identify them accordingly. Similarly, i_* maps standard modules for $(B/J)_0$ to those for B_0 , and we denote them both by $\Delta^0(\gamma)$. Similar remarks apply to costandard modules. Since i_* is a full embedding at the derived category level, it preserves Ext-groups. Therefore, B/J is also a standard Q-Koszul algebra.

Consider next the quasi-hereditary algebra eBe . The module category $eBe\text{-mod}$ is equivalent to the quotient category $B\text{-mod}/B/J\text{-mod}$, a highest weight category with weight poset $\Omega := \Lambda \setminus \Gamma$. All this fits into a standard recollement diagram

$$\begin{array}{ccccc} D^b(B/J\text{-mod}) & \xleftarrow{i^*} & D^b(B\text{-mod}) & \xleftarrow{j_!} & D^b(eBe\text{-mod}) \\ & \xrightarrow{i_*} & & \xrightarrow{j^*} & \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

A similar recollement diagram is obtained by replacing $B\text{-mod}$, $B/J\text{-mod}$, $eBe\text{-mod}$ by $B\text{-grmod}$, $B/J\text{-grmod}$, $eBe\text{-grmod}$, respectively. It is well-known that given any $\omega \in \Omega$, $j^*\Delta(\omega)$ (respectively, $j^*\nabla(\omega)$) is the standard (respectively, costandard) module for $eBe\text{-mod}$ attached to γ . Similarly, $j^*L(\omega)$ is the irreducible eBe -module attached to ω . In addition, $j_!j^*\Delta(\omega) \cong \Delta(\omega)$ and $j_*j^*\nabla(\omega) \cong \nabla(\omega)$. The same holds true for $\Delta^0(\omega)$ and $\nabla^0(\omega)$. Consequently, we see that eBe is itself a standard Q-Koszul algebra.

4. Ext-ALGEBRAS

First, we recall some recent results of Madsen [24], [25], and we very briefly indicate some problems suggested by these results in the context of Q-Koszul algebras. This topic will be discussed further in [37].

Let A be a positively graded finite dimensional algebra, and let T be a finite dimensional tilting module for the algebra A_0 . Then T is regarded as an A -module through the natural map $A \rightarrow A_0$. Assume that A_0 has finite global dimension. Following [24], the algebra A is defined to be T -Koszul provided that $\text{ext}_A^j(T, T\langle i \rangle) \neq 0 \implies i = j$. (Madsen [24] does not require that A be finite dimensional, only finite dimensional in each grade.)

Rather than define a tilting module T for A_0 , we are interested here in the special case in which A_0 is a QHA with weight poset Λ . In this situation, $T = \bigoplus_{\lambda \in \Lambda} T(\lambda)^{\oplus n_\lambda}$, where the n_λ are positive integers, and $T(\lambda)$ is the unique (up to isomorphism) indecomposable A_0 -module of highest weight λ which has both a Δ - and a ∇ -filtration. In other words, T is a full tilting module in the sense of quasi-hereditary algebras; cf. Ringel [41] and especially Donkin [15].

Proposition 4.1. *Assume that A is a Q-Koszul algebra. Then A is T -Koszul for any (full) tilting module T for the QHA A_0 .*

Proof. Since T has a Δ^0 -filtration and a ∇_0 -filtration, if $\text{ext}_A^j(T, T\langle i \rangle) \neq 0$, then, for some $\lambda, \mu \in \Lambda$, $\text{ext}_A^j(\Delta^0(\lambda), \nabla_0(\mu)\langle i \rangle) \neq 0$, so that, by (1.0.2), $i = j$, as required. \square

In [24, Thm. 4.2.1], it is proved that if A is any T -Koszul algebra, then

$$A^\dagger := \text{Ext}_A^\bullet(T, T)^{\text{op}} \quad \text{is } T^*\text{-Koszul.}$$

Here $T^* := \text{Hom}_k(T, k)$, viewed as a left (tilting) module for $A_0^\dagger = \text{End}_A(T, T)^{\text{op}} \cong \text{End}_{A_0}(T, T)^{\text{op}}$. In addition, $A \cong A^{\dagger\dagger}$. Moreover, if A has finite global dimension (as is the case if A is a QHA), then [24, Thm. 4.3.4], applied with $A = \Gamma$ and $A^\dagger = \Lambda$ there, gives an equivalence

$$G_{T^*}^b : D^b(A^\dagger\text{-grmod}) \xrightarrow{\sim} D^b(A\text{-grmod})$$

of triangulated categories.

The relationship of Q- and T-Koszulity needs to be understood better. On the one hand, the very ‘‘Koszul-like’’ quadratic property proved in Theorem 2.3(a) seems to be unknown for general T-Koszul algebras, even those of finite global dimension. On the other hand, all current knowledge of A^\dagger above currently derives from the T-Koszul property, which tells us, at present, only that A^\dagger is T^* -Koszul. In particular, we do not know if A^\dagger is Q-Koszul if A is Q-Koszul, or even if A is standard Q-Koszul. Formally, we ask the following questions.

Questions 4.2. Let A be a Q-Koszul algebra.

- (a) Under what conditions is the algebra A^\dagger also Q-Koszul?
- (b) If A is standard Q-Koszul, under what conditions is A^\dagger also standard Q-Koszul?
- (c) In those cases in which the answer to (b) is positive and A has a Lie theoretic (or geometric) interpretation, is there a corresponding interpretation for A^\dagger ? The classic example is the case of parabolic-singular duality in the category \mathcal{O} for a complex semisimple Lie algebra; cf [3].

Remark 4.3. Suppose that A is standard Q-Koszul. Then using the methods of [8] or [29], the following product formula can be deduced for $n \in \mathbb{N}$, $M \in \mathcal{E}^L$, and $N \in \mathcal{E}^R$,

(4.0.20)

$$\dim \text{Ext}_A^n(M, N) = \sum_{a+b=n} \sum_{\nu \in \Lambda} \dim \text{Ext}_A^a(\Delta(\nu), N) \cdot \dim \text{Ext}_A^b(M, \nabla(\nu)).$$

In particular, the above equation holds for $M = \Delta^0(\lambda)$, $N = \nabla_0(\mu)$ for $\lambda, \mu \in \Lambda$.

We do not prove (4.0.20) here, but refer instead to the paper [37] in preparation. (The methods are similar to those in [8] or [29], working with enriched Grothendieck groups.) The formula (4.0.20) suggests that Question 4.2(b) has a positive answer without any further conditions on A , i. e., if A is standard Q-Koszul, then A^\dagger is always standard Q-Koszul as well. This insight comes from [9]. There, conditions are satisfied, in the context of Kazhdan-Lusztig theory, which guarantee that the homological dual of a (suitably structured) quasi-hereditary algebra is again quasi-hereditary, or even possesses stronger properties. The proof involves a formula like that in (4.0.20) with M and N irreducible.

Part II: An example in characteristic $p = 2$

Some further notation

The paragraphs below briefly describe the setup/notation employed in several previous papers [12], [30], [32], [33], [31], [34], [35], and [36]. This material will be used in the sections below.

Let G be a semisimple, simply connected algebraic group over an algebraically closed field k of positive characteristic p . Fix a maximal torus T , contained in a Borel subgroup B corresponding to the negative roots, etc. We follow the notation of [20] carefully, except, given a dominant weight $\lambda \in X(T)_+$, $\Delta(\lambda)$ (respectively, $\nabla(\lambda)$) is the standard (respectively, cosandard) module of highest weight λ . Thus, $\Delta(\lambda)$ (respectively, $\nabla(\lambda)$) has irreducible head (respectively, socle) $L(\lambda)$. The set $X(T)_+$ is partially ordered by setting $\lambda \leq \mu$ provided $\mu - \lambda$ is a sum of positive roots. We will work with ideals in $X(T)_+$ or in $X_{\text{reg}}(T)_+$ (the set of p -regular dominant weights, given its induced poset structure). If Γ is a finite ideal, in either $X(T)_+$ or $X_{\text{reg}}(T)_+$, there is a QHA algebra $A = A_\Gamma$ such that $A\text{-mod}$ is equivalent to the category of finite dimensional rational G -modules which have composition factors $L(\gamma)$, $\gamma \in \Gamma$. Indeed, we can assume A is an appropriate quotient algebra of the distribution algebra of G .

The algebra A_Γ can also be studied using the quantum enveloping algebra U_ζ at an $\ell(p)$ th root of unity associated to G . Here $\ell(p) = p$ if p is odd, and $\ell(2) = 4$; see (7.0.29) below in §7. There is an appropriate p -modular system (K, \mathcal{O}, k) such that U_ζ is regarded as a K -algebra. In addition, there is a (split) QHA quotient algebra $A' = A'_\Gamma$ of U_ζ such that $A'\text{-mod}$ is equivalent to the category of finite dimensional (type 1, integrable) U_ζ -modules with composition factors $L'(\gamma) := L_\zeta(\gamma)$, $\gamma \in \Gamma$. In addition, there is an \mathcal{O} -order \tilde{A} such that $\tilde{A}_K \cong A'$ and $\tilde{A}_k \cong A$.

Given $\lambda \in \Gamma$, the irreducible A' -module $L_\zeta(\lambda)$ of highest weight λ contains a minimal admissible lattice $\tilde{L}_{\min}(\lambda)$ and a maximal admissible lattice $\tilde{L}_{\max}(\lambda)$. We set $\Delta^{\text{red}}(\lambda) := \tilde{L}_{\min}(\lambda)_k$ and $\nabla^{\text{red}}(\lambda) = \tilde{L}_{\max}(\lambda)_k$. These modules play an important role in the modular representation theory of G . See [12] and the other references above for more discussion.

Define a positively graded \mathcal{O} -order $\text{gr}\tilde{A} = \bigoplus_{n \geq 0} \text{gr}_n \tilde{A}$ by setting

$$\text{gr}_n \tilde{A} = \frac{\tilde{A} \cap \text{rad}^n A'}{\tilde{A} \cap \text{rad}^{n+1} A'},$$

where $\text{rad}^n A' = (\text{rad } A')^n$.³ Similarly, if \tilde{M} is a \tilde{A} -lattice, there is a graded $\text{gr}\tilde{A}$ -lattice $\text{gr}\tilde{M} := \bigoplus_{n \geq 0} \text{gr}_n \tilde{M}$, where $\text{gr}_n \tilde{M} = (\text{rad}^n \tilde{M})/(\text{rad}^{n+1} \tilde{M})$, with $\text{rad}^n \tilde{M} := \tilde{M} \cap (\text{rad}^n \tilde{M}_K)$. Here $\text{rad}^n \tilde{M}_K = (\text{rad } A')^n \tilde{M}_K$. We can then define the (non-negatively) graded algebra by setting

$$(4.0.21) \quad \tilde{\text{gr}} A := (\text{gr}\tilde{A})_k.$$

In addition, for an \tilde{A} -lattice \tilde{M} , put $M := \tilde{M}_k$ and $\tilde{\text{gr}} M := (\text{gr}\tilde{M})_k$.

³By an \mathcal{O} -order (or simply an order if \mathcal{O} is clear) we mean an \mathcal{O} -algebra \tilde{B} which is a free \mathcal{O} -module of finite rank. A \tilde{B} -module \tilde{M} is a lattice, if it is free of finite rank over \mathcal{O} .

The papers cited above contain many properties of the algebras $\widetilde{\text{gr}} A$ and their modules $\widetilde{\text{gr}} M$, as well as alternative definitions for them (usually involving the small quantum enveloping algebra). Some of these results will be mentioned in the final two sections of this paper.

But it is important to observe that the graded algebra $\text{gr} A := \bigoplus_{n \geq 0} \text{rad}^n A / \text{rad}^{n+1} A$ is often different from the algebra $\widetilde{\text{gr}} A$. The algebra $\widetilde{\text{gr}} A$ is generally quasi-hereditary as is its grade 0 part $(\widetilde{\text{gr}} A)_0$, with the same weight poset as $\widetilde{\text{gr}} A$. Indeed, $(\widetilde{\text{gr}} A)_0$ appears to be highly worthy of further study. See Remarks 7.2.

The next two sections work with a slight variation of $\widetilde{\text{gr}} A$, replacing A by a Schur algebra. (In fact, the Schur algebra could be placed in the current context, but we omit the details. In addition, Schur algebras are more familiar to most readers.)

5. TURNER'S COUNTEREXAMPLE.

Given a semisimple, simply connected algebraic group G , the main result in [33] establishes that any standard module $\Delta(\lambda)$ has a Δ^{red} -filtration, provided the characteristic p of the base field k is sufficiently large (depending on the root system of G). However, when p is small this result sometimes fails. In fact, an unpublished counterexample has been shown to us by Will Turner which involves the Schur algebra $S(5, 5)$ when $p = 2$. In this section, we consider Turner's example for $S(5, 5)$ in some detail. In §6, we show that despite the counterexample, the modules $\Delta^{\text{red}}(\lambda)$ do fit into an elegant standard Q-Koszul theory in the case for $S(5, 5)$ and $p = 2$.

Specifically, there is a "forced graded" version $\widetilde{\text{gr}} S$ of $S := S(5, 5)$. This graded algebra is obtained in the same way as the algebra $\widetilde{\text{gr}} A$ above the start of this section, but using the complex q -Schur algebra $S' := S_q(5, 5)$, with $q = -1$.⁴ Then the modules $\Delta^{\text{red}}(\lambda)$ are the standard modules for $(\widetilde{\text{gr}} S)_0$, a quasi-hereditary quotient algebra of S itself. The main result in §6, which is built on the results of this section, shows that $\widetilde{\text{gr}} S$ is standard Q-Koszul. The authors regard this highly non-trivial result in the smallest possible characteristic as quite remarkable. Together with the large prime results mentioned in §7, it inspires the conjectures given there.

The discussion requires some standard partition terminology. For a positive integer r , let $\Lambda^+(r)$ (respectively, $\Lambda(r)$) be the set of partitions (compositions) of r with at most r nonzero parts. Let $\Lambda_{\text{reg}}^+(r) \subset \Lambda^+(r)$ be the 2-regular partitions (i. e., $\lambda \in \Lambda_{\text{reg}}^+(r) \iff$ no part of λ is repeated 2 or more times).⁵ If $\lambda \in \Lambda(5)$, λ^* is the dual partition.

Let k be an algebraically closed field of characteristic 2. Let $R = k\mathfrak{S}_5$ be the group algebra of the symmetric group \mathfrak{S}_5 over k . (This R is obviously not to be confused with the root system which has the same name.) For $\lambda \in \Lambda(5)$, let \mathfrak{S}_λ be the Young subgroup

⁴In the notation of [38] and [5], S' would be denoted $S_{\sqrt{-1}}(5, 5)$. In addition, S' is a homomorphic image of the quantum enveloping algebra for $\mathfrak{gl}_5(\mathbb{C})$ at $\sqrt{-1}$.

⁵The 2-regular partitions should not be confused with the set of 2-regular weights in the sense of alcove geometry.

of \mathfrak{S}_5 defined by λ . The Poincaré polynomial of \mathfrak{S}_λ is defined by $p_{\mathfrak{S}_\lambda}(q) = \sum_{w \in \mathfrak{S}_\lambda} q^{\ell(w)}$. Taking $\lambda = (5)$, $\mathfrak{S}_\lambda = \mathfrak{S}_5$ has Poincaré polynomial

$$p_{\mathfrak{S}_5}(q) = \prod_{i=1}^5 \frac{q^i - 1}{q - 1}.$$

In particular, we will need the fact that

$$(5.0.22) \quad r_{(3,2)}(q) := p_{\mathfrak{S}_5}(q)/p_{\mathfrak{S}_{(3,2)}}(q) = (1 + q^2)(1 + q + q^2 + q^3 + q^4).$$

Let ${}^\lambda\mathfrak{S}_5$ denote the set of distinguished right coset representatives of \mathfrak{S}_λ in \mathfrak{S}_5 . In particular, $r_{(3,2)}(q) = \sum_{d \in (3,2)\mathfrak{S}_5} q^{\ell(d)}$.

Set $T_\lambda := \text{ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_5} k$, the right permutation module for \mathfrak{S}_5 acting on the set of right cosets of the subgroup \mathfrak{S}_λ . If $T = \bigoplus_{\lambda \in \Lambda(5)} T_\lambda$,

$$S(5, 5) := \text{End}_R(T)$$

is, by definition, the Schur algebra of bidegree $(5, 5)$ over k ; see [18]. The category $S(5, 5)\text{-mod}$ is a highest weight category with weight poset $\Lambda^+(5)$. For $\lambda \in \Lambda^+(5)$, let $\Delta(\lambda)$ (respectively, $\nabla(\lambda)$, $L(\lambda)$) be the standard (respectively, costandard, irreducible) $S(5, 5)$ -module indexed by λ .

Let $\text{mod-}R$ be the category of finite dimensional right R -modules. (Recall that $R = k\mathfrak{S}_5$.) It is related to the category $S\text{-mod}$ of finite dimensional left S -modules by the contravariant *diamond functors*:

$$\begin{cases} (-)^\diamond = \text{Hom}_S(-, T) : S\text{-mod} \longrightarrow \text{mod-}R; \\ (-)^\diamond = \text{Hom}_R(-, T) : \text{mod-}R \longrightarrow A\text{-mod}. \end{cases}$$

For $\lambda \in \Lambda^+(5)$, $\Delta(\lambda)^\diamond \cong S_\lambda$, the Specht module for R indexed by λ . The irreducible R -modules are indexed by the set of $\Lambda_{\text{reg}}^+(5)$ of 2-regular partitions; given $\lambda \in \Lambda_{\text{reg}}^+(5)$, D_λ denotes the associated irreducible module. For $\lambda \in \Lambda^+(5)$,

$$(5.0.23) \quad L(\lambda)^\diamond \cong \begin{cases} D_{\lambda^*}, & \lambda^* \in \Lambda_{\text{reg}}^+(5); \\ 0, & \text{otherwise.} \end{cases}$$

(We remark that the description of $L(\lambda)^\diamond$ requires a twist by the sign representation in characteristics different from 2.) Also, for $\lambda, \mu \in \Lambda^+(5)$, $L(\lambda)$ and $L(\mu)$ are in the same block if and only if the partitions λ, μ have the same 2-core.

Because ${}_S T = R^\diamond$ and R_R is a direct summand of T_R , it follows that ${}_S T$ is a projective S -module. It is known that ${}_S T$ is self-dual, so that T is also an injective S -module (and hence a tilting module) and the functor $(-)^\diamond = \text{Hom}_S(-, T)$ is exact.

For $\lambda \in \Lambda^+(5)$, let $X(\lambda)$ be the tilting module for S defined by λ . It has a Δ -filtration with bottom section $\Delta(\lambda)$ and higher sections $\Delta(\mu)$ for partitions $\mu < \lambda$ (in the dominance ordering) having the same 2-core as λ .⁶

A PIM Y in R_R has irreducible socle D_{μ^*} with $\mu^* \in \Lambda^+(5)$. Thus, the corresponding summand Y^\diamond of T is the projective cover $P(\mu)$ of $L(\mu)$. If $\nu \in \Lambda^+(5)$, then

$$[P(\mu) : \Delta(\nu)] = [Y : S_\nu] = [S_\nu : D_{\mu^*}].$$

Taking $\mu = (2^2, 1)$, $\dim D_{\mu^*} = 4$, and D_{μ^*} is the unique non-trivial principal block irreducible module. It follows easily that $P(2^2, 1)$ is filtered by standard modules $\Delta(\nu)$ with $\nu = (2^2, 1)$, $(3, 1^2)$, and $(3, 2)$, each appearing with multiplicity 1. Since we know this PIM for S is an indecomposable tilting module, it must be $X(3, 2)$. Since $X(3, 2) = P(2^2, 1)$ has simple head $L(2^2, 1)$ and is self dual, $X(3, 2)$ has head and socle isomorphic to $L(2^2, 1)$. In particular, this means that

$$(5.0.24) \quad L(2^2, 1) \text{ is the socle of } \Delta(3, 2).$$

On the other hand, $\Delta^{\text{red}}(1^5) \cong L(1^5)$, $\Delta^{\text{red}}(3, 2) \cong L(3, 2)$, and $\Delta^{\text{red}}(3, 1, 1) \cong L(3, 1, 1)$. (The last two isomorphisms are obtained by computing dimensions using versions of Steinberg's tensor product theorem; see the table below.)

Claim: $\Delta^{\text{red}}(2^2, 1) = \Delta(2^2, 1) \not\cong L(2^2, 1)$. (Assuming this fact, it follows that $\Delta(3, 2)$ does not have a Δ^{red} -filtration.)

To check the claim, it will be necessary to use the description of the Δ^{red} -modules from the quantum point of view. This uses the theory of q -Schur algebras as well as their relationship to Hecke algebras by means of quantum Schur-Weyl duality [28], [17].

Let $R' = H(\mathfrak{S}_5)$ be the Hecke algebra over \mathbb{C} of \mathfrak{S}_5 with $q = -1$. It has standard basis τ_w , $w \in \mathfrak{S}_5$, satisfying the familiar relations [17, (1.1)]. For $\lambda \in \Lambda(5)$, let $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} \tau_w$ and set $T'_\lambda = x_\lambda R'$, the q -permutation module defined by λ . In particular, if $\lambda = (5)$, $T'_{(5)} = \mathbb{C}x_{(5)}$ is the one-dimensional index representation of R' . We have, for any λ ,

$$\dim \text{Hom}_{R'}(T'_\lambda, T'_{(5)}) = \dim \text{Hom}_{R'}(T'_{(5)}, T'_\lambda) = 1,$$

by Frobenius reciprocity and the fact that the q -permutation modules are self-dual.

In particular, let $\lambda = (3, 2)$, and consider the nonzero homomorphisms

$$\phi : T'_{(5)} \rightarrow T'_{(3,2)}, \quad x_{(5)} \mapsto \sum_{d \in (3,2)\mathfrak{S}_5} x_{(3,2)} \tau_d$$

and

$$\psi : T'_{(3,2)} \rightarrow T'_{(5)}, \quad x_{(3,2)} h \mapsto x_{(5)} h.$$

⁶More precisely, this observation can be quickly reduced to the case of Young modules for symmetric groups, where it is well-known.

Using (5.0.22),

$$\begin{aligned}\psi \circ \phi &= \sum_{d \in (3,2) \mathfrak{S}_5} q^{\ell(d)} x_{(5)} \\ &= r_{(3,2)}(-1) x_{(5)} = 2x_{(5)}.\end{aligned}$$

For $\lambda \in \Lambda^+(5)$, let S'_λ (respectively, Y'_λ , Y'^{th}_λ) be the corresponding Specht (respectively, Young, twisted Young module) module; see [28, §2]. Then $Y'^{\text{th}}_\lambda \cong Y'^{\Phi}_{\lambda'}$, using the notation of [28, (2.0.4)] with the involution $\Phi : R' \rightarrow R'$ defined in [28, (2.0.3)].

Putting $T' = \bigoplus_{\lambda \in \Lambda(5)} T'_\lambda$, let

$$S' = S_q(5, 5) = S_{-1}(5, 5) := \text{End}_{R'}(T')$$

be the q -Schur algebra of bidegree $(5, 5)$ at $q = -1$. The algebra S' is quasi-hereditary with weight poset $\Lambda^+(5)$, standard objects $\Delta'(\lambda)$ and irreducible objects $L'(\lambda)$ for $\lambda \in \Lambda^+(5)$.

Consider the tilting module $X'(2^2, 1)$ for S' corresponding to the partition $(2^2, 1)$. Then

$$X'(2^2, 1)^\diamond \cong Y'^{\Phi}_{(3,2)}.$$

(To see this, one can use [17, Prop. 7.3(d)] and [10, Lemma 1.5.2].) Now the possible Δ' sections of $X'(2^2, 1)$ are $\Delta'(2^2, 1)$ (with multiplicity 1) and $\Delta'(1^5)$ of an undetermined multiplicity. Therefore, $Y'^{\Phi}_{(3,2)}$ has a filtration with sections $S'_{(2^2,1)} = \Delta'(2^2, 1)^\diamond$ and $S'_{(1^5)} = \Delta'(1^5)^\diamond$. Since S'^{Φ}_λ is dual to S'_λ , and since Y'_λ is self-dual, it follows that $Y'_{(3,2)}$ has a filtration with sections $S'_{(3,2)}$ (with multiplicity 1) and possibly $S'_{(5)}$ (having the same multiplicity as $\Delta'(1^5)$ does in $X'(2^2, 1)$). But $Y'_{(3,2)}$ is an indecomposable summand of $T'_{(3,2)}$. We have already proved that any nonzero homomorphism $T'_{(3,2)} \rightarrow T'_{(5)}$ or $T'_{(5)} \rightarrow T'_{(3,2)}$ splits. It follows that

$$(5.0.25) \quad Y'_{(3,2)} \cong S'_{(3,2)}.$$

Therefore, $X'(2^2, 1) \cong \Delta'(2^2, 1)$. But $X'(2^2, 1)$ is self-dual, so that

$$(5.0.26) \quad \Delta'(2^2, 1) \cong L'(2^2, 1).$$

This forces $\Delta^{\text{red}}(2^2, 1) \cong \Delta(2^2, 1)$. To finish the Claim, it must be checked that $\Delta(2^2, 1)$ is not irreducible. Otherwise, $S_{(2^2,1)} = \Delta(2^2, 1)^\diamond$ is irreducible of dimension 5. But the only possible dimensions of irreducible \mathfrak{S}_5 -modules in characteristic 2 are 1 and 4; see Carlson [4], for example. This completes the proof of the following result.

Proposition 5.1. (Turner) *For the Schur algebra $S(5, 5)$ in characteristic 2, the standard module $\Delta(3, 2)$ does not have a Δ^{red} -filtration.*

6. CONTINUATION: $\widetilde{\text{gr}} S(5, 5)$ IS A STANDARD Q-KOSZUL ALGEBRA IN CHARACTERISTIC 2.

We continue our discussion of $S(5, 5)$, focusing on its principal block A .⁷ The partitions associated to its irreducible modules form a poset Λ , given by the first column of the table below.

Only the trivial module $D_{(5)}$ for $k\mathfrak{S}_5$ has dimension 1, so the other modules all have dimension 4. Applying (5.0.13) and the fact that $S_{(2^2, 1)}$ has dimension 5, gives that $\Delta(2^2, 1)$ has two composition factors $L(2^2, 1)$ and $L(1^5)$, each occurring with multiplicity 1. The additional information in the following table can be readily checked using the Weyl dimension formula and the Steinberg tensor product theorem (both the characteristic 2 version and the quantum version [5]). As mentioned above, the modules $L(\lambda)$ listed are precisely those in the “principal” block for $S(5, 5)$ in characteristic 2 (associated to the determinant representation $\Delta(1^5) = L(1^5)$). Dimensions of the corresponding irreducible modules for the characteristic 0 quantum q -Schur algebra, $q = -1$, are also given. We denote the “principal block” of this q -Schur algebra by A' , and generally decorate with the “prime” symbol objects associated to A' .

λ	$\dim \Delta(\lambda) = \dim \Delta'(\lambda)$	$\dim L'(\lambda)$	$\dim L(\lambda)$
(1^5)	1	1	1
$(2^2, 1)$	75	75	74
$(3, 1^2)$	126	50	50
$(3, 2)$	175	50	50
(5)	126	75	25

Various dimensions of irreducible and standard modules for $S = S(5, 5)$ in characteristic 2 and $S' = S_{-1}(5, 5)$, in characteristic 0

Now we give the matrix D of decomposition numbers $[\Delta(\lambda) : L(\mu)]$. The entries x and y (which will be shown shortly to be equal to 1) are non-negative integer values, momentarily unknown, with $x + y = 2$. In this matrix, all entries and the constraints on x and y can be determined solely from

- (1) The previous table;
- (2) The fact that $[\Delta(\lambda) : L(\mu)] \neq 0$ implies $\mu \leq \lambda$;
- (3) $[\Delta(\lambda) : L(\lambda)] = 1$;
- (4) $[\Delta(\lambda) : L(1^5)] = [S_\lambda : D_{(5)}] \leq \dim S_\lambda \leq 6$.

⁷There is only one other block which is easily handled by ad hoc methods.

	$L(1^5)$	$L(2^2, 1)$	$L(3, 1^2)$	$L(3, 2)$	$L(5)$
$\Delta(1^5)$	1	0	0	0	0
$\Delta(2^2, 1)$	1	1	0	0	0
$\Delta(3, 1^2)$	2	1	1	0	0
$\Delta(3, 2)$	1	1	1	1	0
$\Delta(5)$	1	0	x	y	1

Decomposition matrix D

The first row in the Cartan matrix $C = D^t \cdot D$ has entries 8, 4, 3 + x , 1 + y , 1. No row in C , other than the first row, can have three entries as large as 4 (given the constraint that $x + y = 2$). However, the Cartan matrix corresponding to the PIMs, given in [4] for a block in the algebra H Morita equivalent to $S(5, 5)$, has a row with entries 8, 4, 4, 2, 1 (in some order). For the reader's convenience, this Cartan matrix is given below, with rows and columns as indexed in [4], but in a different order.

Thus, the first row computed above must correspond to the unique row with these entries in [4]. Comparison of rows forces $x = 1$. Thus, $y = 1$. At this point, the matrix $C = D^t \cdot D$ agrees with that in [4] (after a simultaneous reordering of rows and columns) as listed below. No further simultaneous reordering of rows and columns leads to the same 5×5 matrix. Since this matrix is C above, the conversion table below, of partitions to labels in [4], is uniquely determined.

	$L(7)$	$L(2)$	$L(6)$	$L(5)$	$L(4)$
$P(7)$	8	4	4	2	1
$P(2)$	4	3	2	1	0
$P(6)$	4	2	3	2	1
$P(5)$	2	1	2	2	1
$P(4)$	1	0	1	1	1

Cartan matrix C for the principal block for $S(5, 5)$ in characteristic 2, labeling as in [4]

(1^5)	7
$(2^2, 1)$	2
$(3, 1^2)$	6
$(3, 2)$	5
(5)	4

Conversion table from partitions to labeling in [4]

The decomposition matrix D' for the corresponding block of S' can be easily obtained using entries from D , the equality $\Delta'(2^2, 1) = L'(2^2, 1)$, and (5.0.12). It is given below, indexing these modules with the integer labels above. In this terminology, (5.0.12) implies that $[\Delta'(4) : L'(5)] = [P'(5) : \Delta'(4)] = 0$.

	$L'(7)$	$L'(2)$	$L'(6)$	$L'(5)$	$L'(4)$
$\Delta'(7)$	1	0	0	0	0
$\Delta'(2)$	0	1	0	0	0
$\Delta'(6)$	1	1	1	0	0
$\Delta'(5)$	0	1	1	1	0
$\Delta'(4)$	1	0	1	0	1

Decomposition matrix D' for principal block for $S_{-1}(5, 5)$ in characteristic 0

We now describe the radical and socle series for the PIMs corresponding to the irreducible modules $L(7), L(2), L(6), L(5)$ and $L(4)$, as given in [4].

$P(7)$	$P(2)$	$P(6)$	$P(5)$	$P(4)$
$L(7)$	$L(2)$	$L(6)$	$L(5)$	$L(4)$
$L(2), L(6)$	$L(7)$	$L(5), L(7)$	$L(4), L(6)$	$L(5)$
$L(5), L(7), L(7)$	$L(6)$	$L(2), L(4)$	$L(5), L(7)$	$L(6)$
$L(2), L(4), L(6)$	$L(5), L(7)$	$L(5), L(7)$	$L(2), L(6)$	$L(7)$
$L(5), L(7), L(7)$	$L(2)$	$L(6), L(6)$	$L(7)$	
$L(2), L(6)$	$L(7)$	$L(7)$		
$L(7), L(7)$	$L(6)$	$L(2)$		
$L(2), L(6)$	$L(7)$	$L(7)$		
$L(7)$	$L(2)$			

Radical series for PIMs $P(7), P(2), P(6), P(5), P(4)$

Remark 6.1. We record the interesting facts that the radical series table above shows that $\text{gr}S(5, 5)$, the graded algebra obtained from $S(5, 5)$ by grading it through its radical series filtration, is neither Koszul nor quasi-hereditary. First, suppose that $\text{gr}S(5, 5)$ is Koszul and consider the minimal projective resolution of $L(2)$. It begins as $\text{gr}P(7) \xrightarrow{\alpha} P(2) \twoheadrightarrow L(2)$. From the table immediately above, the kernel of α must be an image of $\text{gr}P(2)$ and must also contain $L(4)$ as a composition factor. But $L(4)$ does not appear as a composition factor of $P(2)$, a contradiction. Hence, $\text{gr}S(5, 5)$ is not Koszul. Secondly, suppose that $\text{gr}S(5, 5)$ is QHA. Except for $\text{gr}P(4)$, the head of each graded PIM occurs with multiplicity > 1 in the PIM. It follows that $\text{gr}P(4)$ is a standard module with head $L(4)$. Also, the weight “4” is maximal. Obviously, $\dim \text{Hom}_{\text{gr}S(5,5)}(\text{gr}P(4), \text{gr}P(6)) = 1$. In particular, using the maximality of “4”, the standard module $\text{gr}P(4)$ must appear with multiplicity 1 in a standard module filtration of $\text{gr}P(6)$, and may be taken to occur at the bottom, as a submodule. Also, by (1.0.1),

$$\dim \text{hom}_{\text{gr}S(5,5)}(\text{gr}P(4)\langle m \rangle, \text{gr}P(6)) = 1, \quad \text{for a unique } m.$$

Necessarily, $m = 2$. The resulting graded map must be an injection, since its ungraded version is an injection. Thus, $L(7)\langle 5 \rangle$ lies in the socle of $\text{gr}P(6)$. The table above shows that $\text{gr}P(6)$, viewed as a graded $(\text{gr}A)_0$ -module, has a unique composition factor $L(7)\langle 0 \rangle$, and it occurs in—and exactly as—the graded $(\text{gr}A)_0$ -submodule $\text{gr}P(6)_5$. It follows that the latter $(\text{gr}A)_0$ -submodule is contained in the $\text{gr}A$ -socle of $\text{gr}P(6)$. So, $0 = (\text{gr}A)_1(\text{gr}P(6))_5 = (\text{gr}P(6))_6 \neq 0$, a contradiction. Of course, the algebra $S(5, 5)$ is quasi-hereditary, so the above discussion shows that, in general, the quasi-hereditary property is not preserved under “passing to the radical series grading.” In addition, $S(5, 5)$ is not Koszul. Otherwise, $\text{gr}S(5, 5) \cong S(5, 5)$, a general property of Koszul algebras. However, this isomorphism gives two contradictions, since $\text{gr}S(5, 5)$ has been shown to be neither Koszul nor, unlike $S(5, 5)$, quasi-hereditary.

$P(7)$	$P(2)$	$P(6)$	$P(5)$	$P(4)$
$L(7)$	$L(2)$	$L(6)$	$L(5)$	$L(4)$
$L(2), L(6)$	$L(7)$	$L(7)$	$L(4), L(6)$	$L(5)$
$L(7), L(7)$	$L(6)$	$L(2)$	$L(5), L(7)$	$L(6)$
$L(2), L(6)$	$L(7)$	$L(5), L(7)$	$L(2), L(6)$	$L(7)$
$L(5), L(7), L(7)$	$L(2)$	$L(4), L(6)$	$L(7)$	
$L(2), L(4), L(6)$	$L(5), L(7)$	$L(5), L(7)$		
$L(5), L(7), L(7)$	$L(6)$	$L(2), L(6)$		
$L(2), L(6)$	$L(7)$	$L(7)$		
$L(7)$	$L(2)$			

Socle series for PIMS $P(7), P(2), P(6), P(5), P(4)$

Also, we have the following radical series for the standard modules. It is the same as the socle series.

$\Delta(7)$	$\Delta(2)$	$\Delta(6)$	$\Delta(5)$	$\Delta(4)$
$L(7)$	$L(2)$	$L(6)$	$L(5)$	$L(4)$
	$L(7)$	$L(7)$	$L(6)$	$L(5)$
		$L(2)$	$L(7)$	$L(6)$
		$L(7)$	$L(2)$	$L(7)$

The radical/socle series for standard modules for $S(5, 5)$

The first and second columns are clear from the decomposition matrix D above. The column for $\Delta(6)$ follows by inspecting first the socle series for $P(6)$ and then its radical series. Standard quasi-hereditary theory says that $\Delta(6)$ is a quotient of $P(6)$, and is the unique quotient with the composition factors of $\Delta(6)$ (counting multiplicities). Also, $\Delta(4) = P(4)$. Now consider $\Delta(5)$. Finally, $\Delta(5) \cong P(5)/\Delta(4)$. By (5.0.24), $\Delta(5)$ has

socle $L(2)$. (Note also (from the radical series of $P(7)$) that is no non-trivial extension between $L(4)$ and $L(7)$). It follows that the description of $\Delta(5)$ is as indicated.

Now we begin to discuss the quantum case. First, the table below describes the radical series for the quantum standard modules $\Delta'(\lambda)$.

$\Delta'(7)$	$\Delta'(2)$	$\Delta'(6)$	$\Delta'(5)$	$\Delta'(4)$
$L'(7)$	$L'(2)$	$L'(6)$	$L'(5)$	$L'(4)$
		$L'(7), L'(2)$	$L'(6)$	$L'(6)$
			$L'(2)$	$L'(7)$

Radical=socle series for quantum standard modules for $S_{-1}(5, 5)$ in characteristic 0

To see this, first note that $\Delta'(7) = L'(7)$ and $\Delta'(2) = L'(2)$ from the decomposition matrix. Suppose that $L'(\lambda)$ is a submodule of $\Delta'(\mu)$. Then $L'(\lambda) \cap \tilde{\Delta}(\mu)$ is both a full lattice in $L'(\lambda)$ and a pure submodule of $\tilde{\Delta}(\mu)$. Thus, some composition factor of $\Delta^{\text{red}}(\lambda)$ must appear in the socle of $\Delta(\mu)$. Consequently, the socle of $\Delta'(4)$ must be $L'(7)$ and the socle of $\Delta'(5)$ must be $L'(2)$. Using also the quantum decomposition matrix, we get the columns for $\Delta'(5)$ and $\Delta'(4)$. From this information, $\text{Ext}^1(L'(6), L'(2)) \neq 0 \neq \text{Ext}^1(L'(6), L'(7))$ so the middle column in the table follows.

We next describe the Δ^{red} -modules in the following table which gives their radical and socle series. The ∇_{red} are described by turning the diagrams upside down.

$\Delta^{\text{red}}(7)$	$\Delta^{\text{red}}(2)$	$\Delta^{\text{red}}(6)$	$\Delta^{\text{red}}(5)$	$\Delta^{\text{red}}(4)$
$L(7)$	$L(2)$	$L(6)$	$L(5)$	$L(4)$
	$L(7)$			$L(5)$

Before giving the next table, note there is a natural ‘‘Loewy index’’ $\ell(P'(\lambda), \Delta'(\mu))$ that can be defined for any Δ -filtration section $\Delta(\mu)$ of a PIM $P(\lambda)$. for a quasi-hereditary algebra with weight poset Λ . For simplicity, we consider only the case when $\Delta(\mu)$ appears with multiplicity one as a section of $P(\lambda)$.⁸ Namely, using the ‘‘prime’’ notation here, extend the natural map $\Delta'(\mu) \rightarrow \nabla'(\mu)$ to a map $f : P'(\lambda) \rightarrow \nabla'(\mu)$. The multiplicity one assumption guarantees uniqueness of such an extension. Now define $\ell(P'(\lambda), \nabla'(\mu))$ to be the Loewy length of $f(P'(\lambda)) - 1$. In the table below these Loewy lengths are indicated in the left hand column. They can be computed using the previous table and the natural duality on $S'(5, 5)$.

⁸More generally, define $\ell(P(\lambda), \Delta(\mu))$ to be one less than the maximum Lowey length of $f(P(\lambda))$, with f ranging over all $f \in \text{Hom}(P(\lambda), \nabla(\mu))$ (or just over a basis of the latter space). The definition can be used to define $\ell(M, \Delta(\mu))$ for any finite dimensional module M for the underlying quasi-hereditary algebra.

	$P'(7)$	$P'(2)$	$P'(6)$	$P'(5)$	$P'(4)$
0	$\Delta'(7)$	$\Delta'(2)$	$\Delta'(6)$	$\Delta'(5)$	$\Delta'(4)$
1	$\Delta'(6)$	$\Delta'(6)$	$\Delta'(5), \Delta'(4)$		
2	$\Delta'(4)$	$\Delta'(5)$			

According to [42], the algebra A' is standard Koszul.⁹ In particular, $\text{gr}A' \cong A'$ is a quasi-hereditary algebra with $\text{gr}P'(\lambda) \cong P'(\lambda)$ and $\text{gr}\Delta'(\lambda) \cong \Delta'(\lambda)$ as graded modules. Each PIM $\text{gr}P'(\lambda)$ has a filtration by shifted standard modules $\text{gr}\Delta'(\mu)\langle s \rangle$, for $s \geq 0$. The multiplicity

$$[\text{gr}P'(\lambda) : \text{gr}\Delta'(\mu)\langle s \rangle] = \dim \text{hom}_{\text{gr}A'}(\text{gr}P'(\lambda), \text{gr}^\circ \nabla'(\mu)\langle s \rangle).$$

That is, this number is precisely the multiplicities of $L'(\lambda)$ in the $-s$ th socle layer of $\text{gr}^\circ \nabla'(\mu)$, or equivalently, in the $(-s)$ th socle layer of $\nabla'(\mu)$ itself. In our case, where $[P'(\lambda) : \Delta'(\mu)] \leq 1$, this multiplicity is 1 if $\ell(P'(\lambda), \Delta'(\mu)) = s$, and 0 otherwise. Thus, the table above may be reinterpreted as giving the required graded multiplicities. We repeat it for emphasis, with the left hand column now giving graded multiplicity information.

	$\text{gr}P'(7)$	$\text{gr}P'(2)$	$\text{gr}P'(6)$	$\text{gr}P'(5)$	$\text{gr}P'(4)$
0	$\text{gr}\Delta'(7)$	$\text{gr}\Delta'(2)$	$\text{gr}\Delta'(6)$	$\text{gr}\Delta'(5)$	$\text{gr}\Delta'(4)$
1	$\text{gr}\Delta'(6)\langle 1 \rangle$	$\text{gr}\Delta'(6)\langle 1 \rangle$	$\text{gr}\Delta'(5)\langle 1 \rangle \oplus \text{gr}\Delta'(4)\langle 1 \rangle$		
2	$\text{gr}\Delta'(4)\langle 2 \rangle$	$\text{gr}\Delta'(5)\langle 2 \rangle$			

Using the standard Koszulity of A' and the criterion [31, Thm. 4.17], it can be shown that $\text{gr}\tilde{S}(5, 5)$ is a graded integral quasi-hereditary algebra, in the sense of [7].¹⁰ It can also be shown that $\tilde{\text{gr}}A$ has an anti-involution inherited from an integral form of A and which preserves grades. Moreover, composition with the usual linear dual functor, induces a duality $X \mapsto X^\circ$ of A -grmod and $\tilde{\text{gr}}A$ -mod, which irreducible modules of pure grade 0, and sending $L\langle r \rangle$ to $L\langle -r \rangle$. If X is any A -module, let $\tilde{\text{gr}}^\circ X$ denote the graded module $(\tilde{\text{gr}}(X^\circ))^\circ$. Thus, $\tilde{\text{gr}}^\circ \nabla(\lambda)$ is obtained by dualizing $\tilde{\text{gr}}\Delta(\lambda) = \tilde{\text{gr}}(\nabla(\lambda)^\circ)$. Many of the tables given above and below have natural duals which we use without comment.

Now we can base change to k , to see that $\tilde{\text{gr}}S(5, 5)$ is a graded quasi-hereditary algebra, with graded PIMS having $\tilde{\text{gr}}\Delta$ -filtrations described by the table below.

⁹A Koszul algebra is standard Koszul if it is quasi-hereditary and if its standard modules have a “linear” projective resolution. Linear here means that the terms in cohomological degree $-n$ are generated in grade n . See [26].

¹⁰Since $\text{gr}A'$ is a QHA here, the criterion requires only that each module $\text{gr}\tilde{\Delta}(\lambda)$ have an irreducible head for each $\lambda \in \Lambda$. Equivalently, it must be shown that each $\tilde{\text{gr}}\Delta(\lambda)$ has an irreducible head. This is also equivalent to the surjectivity of the natural map $\tilde{\text{gr}}P(\lambda) \rightarrow \tilde{\text{gr}}\Delta(\lambda)$, which is equivalent to the term-by-term surjectivity of each of the filtration terms used in forming these graded modules. Surjectivity of the 0th and 1st filtration term is automatic, and this fact alone gives a simple head for $\tilde{\text{gr}}\Delta(2)$, $\tilde{\text{gr}}\Delta(7)$, and $\tilde{\text{gr}}\Delta(6)$. Surjectivity for $\tilde{\text{gr}}\Delta(4)$ is automatic, since $P(4) = \Delta(4)$. The relevant filtrations for $P(5)$ and $\Delta(5)$, the last case, can be analyzed using the splitting $\tilde{P}(5)_K \cong \Delta'(4) \oplus \Delta'(5)$.

	$\widetilde{\text{gr}} P(7)$	$\widetilde{\text{gr}} P(2)$	$\widetilde{\text{gr}} P(6)$	$\widetilde{\text{gr}} P(5)$	$\widetilde{\text{gr}} P(4)$
0	$\begin{smallmatrix} \widetilde{\text{gr}} \Delta(7) \\ \widetilde{\text{gr}} \Delta(2) \end{smallmatrix}$	$\widetilde{\text{gr}} \Delta(2)$	$\widetilde{\text{gr}} \Delta(6)$	$\begin{smallmatrix} \widetilde{\text{gr}} \Delta(5) \\ \widetilde{\text{gr}} \Delta(4) \end{smallmatrix}$	$\widetilde{\text{gr}} \Delta(4)$
1	$\widetilde{\text{gr}} \Delta(6)\langle 1 \rangle \oplus \widetilde{\text{gr}} \Delta(6)\langle 1 \rangle$	$\widetilde{\text{gr}} \Delta(6)\langle 1 \rangle$	$\begin{smallmatrix} \widetilde{\text{gr}} \Delta(5)\langle 1 \rangle \\ \widetilde{\text{gr}} \Delta(4)\langle 1 \rangle \end{smallmatrix}$		
2	$\begin{smallmatrix} \widetilde{\text{gr}} \Delta(5)\langle 2 \rangle \\ \widetilde{\text{gr}} \Delta(4)\langle 2 \rangle \end{smallmatrix}$	$\widetilde{\text{gr}} \Delta(5)\langle 2 \rangle$			

We can also obtain the following resolutions:

$$\begin{cases} 0 \rightarrow \widetilde{\text{gr}} P(4) \rightarrow \widetilde{\text{gr}} \Delta(4) \rightarrow 0 \\ 0 \rightarrow \widetilde{\text{gr}} P(4) \rightarrow \widetilde{\text{gr}} P(5) \rightarrow \widetilde{\text{gr}} \Delta(5) \rightarrow 0 \\ 0 \rightarrow \widetilde{\text{gr}} P(5)\langle 1 \rangle \rightarrow \widetilde{\text{gr}} P(6) \rightarrow \widetilde{\text{gr}} \Delta(6) \rightarrow 0 \\ 0 \rightarrow \widetilde{\text{gr}} P(4)\langle 2 \rangle \rightarrow \widetilde{\text{gr}} P(6)\langle 1 \rangle \rightarrow \widetilde{\text{gr}} P(2) \rightarrow \widetilde{\text{gr}} \Delta(2) \rightarrow 0 \\ 0 \rightarrow \widetilde{\text{gr}} P(4)\langle 2 \rangle \rightarrow \widetilde{\text{gr}} P(5)\langle 2 \rangle \rightarrow \widetilde{\text{gr}} P(6)\langle 1 \rangle \oplus \widetilde{\text{gr}} P(2) \rightarrow \widetilde{\text{gr}} P(7) \rightarrow \widetilde{\text{gr}} \Delta(7) \rightarrow 0. \end{cases}$$

These are obtained from the above table, together with examination of various spaces $\text{hom}(\widetilde{\text{gr}} P(\lambda), \widetilde{\text{gr}}^\diamond \nabla(\mu)\langle s \rangle)$ for various $\lambda, \mu \in \Lambda$.¹¹ For example, consider the more detailed structure of $\widetilde{\text{gr}} P(7)$ which the aim of providing a graded projective cover of the kernel M of the homomorphism $\widetilde{\text{gr}} P(7) \twoheadrightarrow \widetilde{\text{gr}} \Delta(7)$. From the table $\widetilde{\text{gr}} \Delta(4)\langle 2 \rangle$ appears once in a graded $\widetilde{\text{gr}} \Delta$ -filtration of $\widetilde{\text{gr}} P(\lambda)$ appearing as a submodule. Consequently,

$$\dim \text{hom}(\widetilde{\text{gr}} P(7), \widetilde{\text{gr}}^\diamond \nabla(4)\langle 2 \rangle) = 1.$$

The image of a non-zero element $f \in \text{hom}(\widetilde{\text{gr}} P(7), \widetilde{\text{gr}}^\diamond \nabla(4)\langle 2 \rangle)$ must clearly be all of $\widetilde{\text{gr}}^\diamond \nabla(4)\langle 2 \rangle$, since the latter has head $L(7)$. Restricting f to M picks out a filtered submodule

$$\begin{bmatrix} \widetilde{\text{gr}} \Delta(6)\langle 1 \rangle \\ \widetilde{\text{gr}} \Delta(5)\langle 2 \rangle \\ \widetilde{\text{gr}} \Delta(4)\langle 2 \rangle \end{bmatrix}$$

with image

$$\text{rad } \widetilde{\text{gr}}^\diamond \nabla(4)\langle 2 \rangle = \begin{bmatrix} L(6)\langle 1 \rangle \\ L(5)\langle 2 \rangle \\ L(4)\langle 2 \rangle \end{bmatrix}$$

It follows easily that the left hand module is indecomposable with a simple head $L(6)\langle 1 \rangle$. Consequently, it must be isomorphic to $\widetilde{\text{gr}} P(6)\langle 1 \rangle$. The quotient of M by this submodule has a filtration

$$\begin{bmatrix} \widetilde{\text{gr}} P(7) \\ \widetilde{\text{gr}} \Delta(2) \\ \widetilde{\text{gr}} \Delta(6)\langle 1 \rangle. \end{bmatrix}$$

¹¹Here $\widetilde{\text{gr}}^\diamond \nabla(\mu)$ is the co-standard module for $\widetilde{\text{gr}} A$ associated to μ .

The space $\text{hom}(\widetilde{\text{gr}} P(7), \widetilde{\text{gr}} \nabla^\diamond(6)\langle 1 \rangle)$ is 2-dimensional. One of its basis elements has already been “used” in the embedding $\widetilde{\text{gr}} P(6)\langle 1 \rangle / N \subseteq \widetilde{\text{gr}} P(7) / N$ with $N = \frac{\widetilde{\text{gr}} \Delta(5)\langle 2 \rangle}{\widetilde{\text{gr}} \Delta(4)\langle 2 \rangle}$. Any second basis element of this hom-space must have image

$$\begin{array}{c} L(7) \\ L(2) \\ L(7) \\ L(6)\langle 1 \rangle \end{array}$$

It follows $M / \widetilde{\text{gr}} P(6)\langle 1 \rangle$ is a homomorphic image of $\widetilde{\text{gr}} P(2)$. Considering filtration multiplicities, the $(\widetilde{\text{gr}} \Delta$ -filtered) kernel of $\widetilde{\text{gr}} P(6)\langle 1 \rangle \rightarrow \widetilde{\text{gr}} P(2) \rightarrow M \rightarrow 0$ must be $\widetilde{\text{gr}} \Delta(5)$. The rest of the resolution is easy.

Theorem 6.2. *Let A be the principal block of $S(5, 5)$ for $p = 2$. Then $\widetilde{\text{gr}} A$ is standard Q-Koszul.*

Proof. The resolutions above can be used to compute $\text{ext}_{\widetilde{\text{gr}} A}^n(\widetilde{\text{gr}} \Delta(\mu), \nabla_{\text{red}}(\lambda)\langle r \rangle)$. For example, we show $\text{ext}_{\widetilde{\text{gr}} A}^1(\widetilde{\text{gr}} \Delta(7), \nabla_{\text{red}}(2)) = 0$. Here the space $\text{hom}(\widetilde{\text{gr}} P(2), \nabla_{\text{red}}(2))$ of 1-cocycles are also 1-coboundaries:

$$\begin{aligned} \text{hom}(\widetilde{\text{gr}} P(2), \Delta^{\text{red}}(2)) &\cong \text{hom}(\Delta^{\text{red}}(2), \nabla_{\text{red}}(2)) \\ &\cong \text{hom}\left(\frac{\Delta^{\text{red}}(7)}{\Delta^{\text{red}}(2)}, \nabla_{\text{red}}(2)\right) \\ &\cong \text{hom}(P_0(7), \nabla_{\text{red}}(2)) \\ &\cong \text{hom}(\widetilde{\text{gr}} P(7), \nabla_{\text{red}}(2)). \end{aligned}$$

The other cases are checked similarly. □

Remarks 6.3. (a) Any dominant weight λ for a semisimple, simply connected algebraic group G can be uniquely written $\lambda = \lambda_0 + p\lambda_1$, where λ_0 is p -restricted and λ_1 is dominant. Put $\Delta^p(\lambda) := L(\lambda_0) \otimes \Delta(\lambda_1)^{[p]}$. In 1980, Jantzen [19] raised question whether every standard module $\Delta(\lambda)$ for a semisimple group G has a Δ^p -filtration, i. e., a filtration with sections $\Delta^p(\mu)$. See also [1]. While $\Delta(3, 2)$ (as discussed in §5.1) does not have an Δ^{red} -filtration, it does have a Δ^p -filtration. However, we know of no analogue of Q-Koszul algebras involving semisimple groups with uses the Δ^p -modules in place of the Δ^{red} -modules.

(b) It is especially interesting to compare the resolutions for $\widetilde{\text{gr}} \Delta(i)$ above with the corresponding resolutions at the quantum level (i. e., for the $\text{gr} \Delta'(i)$). The latter resolutions can be easily be obtained from those of the integral versions of the $\widetilde{\text{gr}} \Delta$'s and base change,

together with the tables for the various $\text{gr}P'$ -modules. We give them below:

$$\begin{cases} 0 \rightarrow \text{gr}P'(4) \rightarrow \text{gr}\Delta'(4) \rightarrow 0 \\ 0 \rightarrow \text{gr}P'(4) \rightarrow \text{gr}P'(5) \rightarrow \text{gr}\Delta'(5) \rightarrow 0 \\ 0 \rightarrow \text{gr}P'(4)\langle 1 \rangle \oplus \text{gr}P'(5)\langle 1 \rangle \rightarrow \text{gr}P'(6) \rightarrow \text{gr}\Delta'(6) \rightarrow 0 \\ 0 \rightarrow \text{gr}P'(4)\langle 2 \rangle \rightarrow \text{gr}P'(6)\langle 1 \rangle \rightarrow \text{gr}P'(2) \rightarrow \text{gr}\Delta'(2) \rightarrow 0 \\ 0 \rightarrow \text{gr}P'(5)\langle 2 \rangle \rightarrow \text{gr}P'(6)\langle 1 \rangle \rightarrow \text{gr}P'(7) \rightarrow \text{gr}\Delta'(7) \rightarrow 0. \end{cases}$$

In spite of the differences with the resolutions for the $\tilde{\text{gr}}\Delta$ -modules given above, the reader may check in each case that they lead to

$$(6.0.27) \quad \dim \text{ext}_{\tilde{\text{gr}}A}^n(\tilde{\text{gr}}\Delta(\mu), \nabla_{\text{red}}(\lambda)\langle m \rangle) = \dim \text{ext}_{\text{gr}A'}^n(\text{gr}\Delta'(\mu), L'(\lambda)\langle m \rangle).$$

Important in verifying this is the fact that each $(\tilde{\text{gr}}P(\lambda))_0 = P_0(\lambda)$ does have a Δ^{red} -filtration. For example, $\text{hom}_{\tilde{\text{gr}}A}(P(5)\langle 1 \rangle, \nabla_{\text{red}}(4)\langle 1 \rangle)$ is 1-dimensional, since $(\tilde{\text{gr}}P(5))_0 = P_0(5) = \frac{\Delta^{\text{red}}(5)}{\Delta^{\text{red}}(4)}$. This discussion and others like it (such as the sample calculation in the proof of Theorem 6.2) go through, even though $\tilde{\text{gr}}P(\lambda)_m$, $m > 0$, may not have a Δ^{red} -filtration. For example, $\tilde{\text{gr}}P(2)_4$ does not have a Δ^{red} -filtration. In spite of the latter anomaly, we still have the nice equality (6.0.27) and Theorem 6.2. This suggests it is more important to have Δ^{red} -filtrations at the “top.” Also, (6.0.27) has influenced a conjecture in the next section.

Part III: Conjectures

7. SOME CONJECTURES

Let R be a classical finite root system, which we temporarily assume is irreducible. Let $D = 1$ (respectively, 2; 3) if R has type A_n, D_n, E_6, E_7, E_8 (respectively, $B_n, C_n, F_4; G_2$). Let $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ be the Weyl weight. Define $g = (\rho, \theta_r^\vee)$, where θ_r is the maximal root in R^+ . Let $\widehat{\mathfrak{g}}$ be the (infinite dimensional) untwisted affine Lie algebra associated to R , and let

$$\widetilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}] = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}c$$

be its commutator subalgebra. For any $\kappa \in \mathbb{Q}$, consider the category \mathcal{O}_κ of $\widetilde{\mathfrak{g}}$ -modules satisfying certain natural properties (especially, that the central element c acts as multiplication by κ). We do not list these here, but refer instead to [43, p. 270].

Kazhdan-Lusztig have defined, for a positive integer ℓ , a functor

$$(7.0.28) \quad F_\ell : \mathcal{O}_{-(\ell/2D)-g} \longrightarrow \mathcal{Q}_\ell.$$

Here \mathcal{Q}_ℓ is the category of integrable, type 1 modules for the Lusztig quantum enveloping algebra corresponding to R at a primitive ℓ th root of 1. The functor F_ℓ is discussed in

[43], whose treatment we largely follow. We will be interested in the case in which ℓ is associated to a prime integer p by the formula

$$(7.0.29) \quad \ell = \ell(p) = \begin{cases} p & \text{if } p \text{ is odd;} \\ 4 & \text{if } p = 2. \end{cases}$$

Definition 7.1. If R is an irreducible root system, a prime p is KL-good provided F_ℓ is an equivalence for $\ell = \ell(p)$. Also, we assume $p \neq 2$ if R has type B_n , C_n , or F_4 , and $p \neq 3$ if R has type G_2 . More generally, if R is any finite root system, then a prime p is KL-good if it is KL-good for each irreducible component of R .

If R is of type A_n , every prime is KL-good. If R has type D_n , then every odd prime is KL-good, and $p = 2$ is also KL-good if n is even. Finally, in any type, if $p > h$, the Coxeter number, then p is KL-good. See [43, Rem. 7.3] for more details.¹²

We will not make use of the Kazhdan-Lusztig correspondence F_ℓ in the discussion below. However, it motivates the restrictions on p in the conjectures that follow. We will discuss this motivation later in this section after Conjecture IIb. In what follows, we make use of the notation introduced at the end of Section 3.

Conjecture I: Assume that G is a semisimple, simply connected algebraic group, defined and split over \mathbb{F}_p , p a prime. Assume p is KL-good for the root system R of G . Let Γ be a finite ideal in $X(T)_+$, and form the QHA algebra $A := A_\Gamma$. Then the graded algebra $\tilde{\text{gr}} A$ is standard Q-Koszul.

We also expect that the \mathcal{O} -order $\text{gr} \tilde{A} = \text{gr} \tilde{A}_\Gamma$ is integral quasi-hereditary (as defined in [7]). In fact, this is likely to be a key step in showing that $\tilde{\text{gr}} A = (\text{gr} \tilde{A})_k$ is quasi-hereditary, an essential ingredient for the standard Q-Koszul property. If $p \geq 2h - 2$ and if Γ consists of p -regular blocks, then [31] establishes that $\text{gr} \tilde{A}$ is integral quasi-hereditary.

We also mention that when $p \geq 2h - 2$ is odd and when the Lusztig character formula holds, then Conjecture I is proved in [35, Thm. 3.7] in the p -regular weight case. The conjecture itself has no such restrictions. In fact, Section 4 proves the conjecture for A equal to the Schur algebra $S(5, 5)$ when $p = 2$. All applicable conjectures in this section have been similarly checked in that case, based on the results developed in §6, through full proofs have not always been included there.

Conjecture II: Continue to assume the hypotheses and notation of Conjecture I (so, in particular, p is KL-good). Let $A' = A'_\Gamma$ be the quasi-hereditary quotient algebra of the

¹²The only cases in which $p = 2$ is known to be KL-good are type A_n and type D_{2n} . All odd primes are known to be KL-good for simply laced classical root systems (A_n and D_n). It would be good to know if this fact remains true in the non-simply laced classical cases (B_n and C_n), or at least have a bound independent of the root system.

quantum enveloping algebra U_ζ at a primitive $\ell(p)$ th-root of unity associated to the ideal Γ . Then ,

$$\begin{cases} (1) & \dim \text{Ext}_{A'}^n(\Delta'(\lambda), L'(\mu)) = \dim \text{Ext}_A^n(\Delta(\lambda), \nabla_{\text{red}}(\mu)), \\ (2) & \dim \text{Ext}_{A'}^n(L'(\lambda), \nabla'(\mu)) = \dim \text{Ext}_A^n(\Delta^{\text{red}}(\lambda), \nabla(\mu)) \\ (3) & \dim \text{Ext}_{A'}^n(L'(\lambda), L'(\mu)) = \dim \text{Ext}_A^n(\Delta^{\text{red}}(\lambda), \nabla_{\text{red}}(\mu)) \end{cases}, \quad \forall \lambda, \mu \in \Lambda, \forall n$$

In the above expressions, the terms $\text{Ext}_{A'}^n$ (respectively, Ext_A^n) on the left (respectively, right) can be replaced by $\text{Ext}_{U_\zeta}^n$ (respectively, Ext_G^n); see [16], for example. When $p > h$ and the Lusztig character formula holds for restricted dominant weights, then Conjecture II is proved for p -regular weights in [12, Thm. 5.4]. Some interesting cases of Conjecture II can be similarly proved assuming only $p > h$: Conjecture II(3) holds for all weights $\lambda, \mu \in pX(T)_+$. Conjecture II(1) (respectively, Conjecture II(2)) holds for all $\mu \in pX(T)_+$ (respectively, $\lambda \in pX(T)_+$ and all λ (respectively, μ) provided only that $p > h$. Note that λ and μ are p -regular in this case. This follows from [12, Thm. 5.4 and §4].

Related to this conjecture are the following two conjectures:

Conjecture IIa *Under the hypothesis of Conjecture II, we have*

$$\begin{cases} (1) & \dim \text{Ext}_A^n(\Delta(\lambda), \nabla_{\text{red}}(\mu)) = \dim \text{Ext}_{\widetilde{\text{gr}} A}^n(\widetilde{\text{gr}} \Delta(\lambda), \nabla_{\text{red}}(\mu)) \\ (2) & \dim \text{Ext}_A^n(\Delta^{\text{red}}(\mu), \nabla(\lambda)) = \dim \text{Ext}_{\widetilde{\text{gr}} A}^n(\Delta^{\text{red}}(\mu), \widetilde{\text{gr}} \diamond \nabla(\lambda)) \\ (3) & \dim \text{Ext}_A^n(\Delta^{\text{red}}(\mu), \nabla_{\text{red}}(\lambda)) = \dim \text{Ext}_{\widetilde{\text{gr}} A}^n(\Delta^{\text{red}}(\mu), \nabla_{\text{red}}(\lambda)). \end{cases}$$

for all $\lambda, \mu \in \Gamma$.

In part (2) above, $\widetilde{\text{gr}} \diamond \nabla(\lambda)$ denotes the costandard module corresponding to λ in the highest weight category $\widetilde{\text{gr}} A$. It has a natural graded structure, concentrated in non-positive grades, with $\nabla_{\text{red}}(\lambda)$ its grade 0 term. Under the assumptions that λ, μ are p -regular and $p \geq 2h - 2$ is an odd prime, parts (1) and (2) of Conjecture II(a) are proved in [36, Thm. 6.6], while part (3) is proved in [36, Thm. 5.3(b)].

Conjecture IIb: *Under the hypothesis of Conjecture II, we have*

$$\begin{cases} (1) & \dim \text{Ext}_{A'}^n(\Delta'(\lambda), L'(\mu)) = \dim \text{Ext}_{\text{gr} A'}^n(\text{gr} \Delta'(\lambda), L'(\mu)) \\ (2) & \dim \text{Ext}_{A'}^n(L'(\lambda), \nabla'(\mu)) = \dim \text{Ext}_{\text{gr} A'}^n(L'(\lambda), \text{gr} \diamond \nabla'(\mu)), \\ (3) & \dim \text{Ext}_{A'}^n(L'(\lambda), L'(\mu)) = \dim \text{Ext}_{\text{gr} A'}^n(\text{gr} L'(\lambda), L'(\mu)) \end{cases}$$

for all $\lambda, \mu \in \Gamma$ and all $n \in \mathbb{N}$. Also, $\text{gr} A'$ is a standard Koszul algebra.

In fact, it also can be conjectured that the algebra A' itself is standard Koszul.¹³ In that case, Conjecture IIb would follow immediately. In type A_n , this has been proved in [42], using, among other things, the Kazhdan-Lusztig correspondence (7.0.28). It seems likely

¹³A standard Koszul algebra A is a Koszul algebra which is QHA, Koszul, and such that the standard (respectively, costandard) modules linear “linear” (respectively, “colinear”). In other words, A is a standard Q-Koszul algebra in which A_0 is semisimple.

that these methods should extend to all types, as long as the Kazhdan-Lusztig correspondence is an equivalence—in particular, when p is KL-good.

Regarding Conjecture IIb as stated, the authors have proved that $\text{gr} A'$ is standard Koszul in the p -regular case in [32] when $p > h$.¹⁴ Some weaker results, describing semisimple filtrations of standard modules, were proved for p -singular weights in [30], sometimes for small p , provided the Kazhdan-Lusztig functor $F_{\ell(p)}$ is an equivalence.

The next conjecture gives new calculations of Ext-group dimensions in singular weight cases. Let $\mathbb{E} = \mathbb{R} \otimes X(T)$ be the Euclidean space associated to the affine Weyl group W_p . Thus, for $\alpha \in R$, $r \in \mathbb{Z}$, $s_{\alpha, rp} : \mathbb{E} \rightarrow \mathbb{E}$ is the reflection defined by $s_{\alpha, rp}(x) = x - [(x, \alpha^\vee)t - pr]\alpha$. Then W_p is generated by the $s_{\alpha, pr}$, and it is, in fact, a Coxeter group with simple reflections $S := \{s_\alpha \equiv s_{\alpha, 0}\}_{\alpha \in S} \cup \{s_{\alpha_0, -p}\}$.

Let

$$\overline{C^-} = \{x \in \mathbb{E} \mid 1 \leq (x + \rho, \alpha^\vee) \leq p\}$$

be the closed “anti-dominant” alcove for W_p . For $\lambda \in X(T)$, there exists a unique $\lambda^- \in \overline{C^-}$ which is W_p -conjugate to λ under the dot action of W_p on \mathbb{E} . Define $\bar{w} \in W_p$ to be the unique element of shortest length in W_p such that $\lambda = \bar{w} \cdot \lambda^-$. Alternatively, if W_I denotes the stabilizer (under the dot action) in W_p of an element $\lambda^- \in \overline{C^-}$, and if $w \in W_p$, then \bar{w} denotes the distinguished (i. e., smallest length) left coset representative for the left coset wW_I .

Let $\mathcal{Z} := \mathbb{Z}[t, t^{-1}]$ be the algebra of Laurent polynomials. Let $f \mapsto \bar{f}$ be the automorphism of \mathcal{Z} which sends t to t^{-1} . Given $x, y \in W_p$, let $P_{x,y} \in \mathcal{Z}$ be the Kazhdan-Lusztig polynomial associated to the pair x, y . It is known that $P_{x,y}$ is a polynomial in $q := t^2$. Also, define

$$P_{\bar{y}, \bar{w}}^{\text{sing}}(t) := \sum_{x \in W_I, \bar{y}x \leq \bar{w}} (-1)^{\ell(x)} P_{\bar{y}x, \bar{w}}(t),$$

As with $P_{x,y}$, $P_{\bar{y}, \bar{w}}^{\text{sing}}(t)$ is also a polynomial in t^2 . We have the following conjecture. We continue to assume the hypotheses and notation of Conjecture II. In particular, A' is a quotient of U_ζ , where ζ is an $\ell = \ell(p)$ th primitive root of unity with p KL-good. The conjecture likely holds as well for other values of ℓ , not associated to any prime—possibly all $\ell \geq 1$ if R is simply laced. See [23, Conj. 2.3] which conjectures some version of the Kazhdan-Lusztig correspondence works in these cases.

Conjecture III: Let Γ be a finite poset in $X(T)_+$. Write $\lambda = \bar{w} \cdot \lambda^-$ as above. If $\mu \in \Gamma$ has the form $\bar{y} \cdot \lambda^-$ for some distinguished left coset representative \bar{y} of W_I , then

$$\sum_{n \geq 0} \dim \text{Ext}_{A'}^n(\Delta'(\mu), L'(\lambda)) t^n = t^{\ell(\bar{w}) - \ell(\bar{y})} \bar{P}_{\bar{y}, \bar{w}}^{\text{sing}}(t).$$

¹⁴In fact the results of [32], in conjunction with the Ext-formulas given in [8] that hold in the presence of “Kazhdan-Lusztig theories,” are sufficient to establish all parts of Conjecture IIb in the p -regular case, assuming that $p > h$.

(If μ does not have the form $\bar{y} \cdot \lambda^-$, then all the groups $\text{Ext}_{A'}^n(\Delta'(\mu), L'(\lambda)) = 0$, as is well known.)

In the above expression, we could replace A' by U_ζ , where ζ is a primitive $\ell(p)$ of unity. The polynomials $P_{\bar{y}, \bar{w}}^{\text{sing}}$ identify with one class of “parabolic Kazhdan-Lusztig polynomials,” as introduced by Deodhar [14] with a different notation. For further discussion, with a different application and yet another notation, see [21, Prop. 2.4, Cor. 4.1]. In particular, it is shown these polynomials always have non-negative coefficients.

The validity of Conjecture III would explicitly calculate the dimensions of Ext-groups between standard modules and irreducible modules for A' . Then the validity of Conjecture II turns this into a similar calculation for the algebraic group G in positive characteristic.

Conjecture III also has consequences for $\text{Ext}_{A'}$ between irreducible modules, assuming that Conjectures IIb also holds. Thus, $\text{gr} A'$ is then standard Koszul, and one obtains a product formula like that in (4.0.20) at the level of the QHA $\text{gr} A'$. Using Conjecture IIb again gives a calculation of $\text{Ext}_{A'}^\bullet$ -groups between irreducible A' -modules, once the dimensions of the groups $\text{Ext}_{A'}^a(\Delta'(\nu), L'(\mu))$ and $\text{Ext}_{A'}^b(L'(\lambda), \nabla'(\nu))$ can be determined. But Conjecture III calculates these dimensions in terms of Kazhdan-Lusztig coefficients. As in type A , one might expect that A' itself is standard Koszul (which would simplify the above discussion).

Once the $\text{Ext}_{A'}$ -groups in the above paragraph have been calculated, the corresponding $\text{Ext}_A^\bullet = \text{Ext}_G^\bullet$ -groups can be calculated, if Conjecture II also holds. Specifically, the dimensions of all groups

$$\text{Ext}_G^\bullet(\Delta^{\text{red}}(\lambda), \nabla_{\text{red}}(\mu)), \quad \text{Ext}_G^\bullet(\Delta(\lambda), \nabla_{\text{red}}(\mu)), \quad \text{Ext}_G^\bullet(\nabla_{\text{red}}(\lambda), \nabla(\mu))$$

for all $\lambda, \mu \in X(T)_+$ can be calculated explicitly in terms of Kazhdan-Lusztig polynomials and their coefficients. Miraculously, all these predicted dimensions are correct for $A = S(5, 5)$ and $p = 2$.

Remarks 7.2. (a) Conjecture II is a consequence of Conjectures IIa and IIb, in the presence of Conjecture I. This is obtained by proving, using the standard Koszulity of $\text{gr} A'$ and the standard Q-Koszulity of $\tilde{\text{gr}} A$, that

$$(7.0.30) \quad \dim \text{ext}_{\text{gr} A'}^n(\text{gr} \Delta'(\lambda), L'(\mu) \langle n \rangle) = \dim \text{ext}_{\tilde{\text{gr}} A}^n(\tilde{\text{gr}} \Delta(\lambda), \nabla_{\text{red}}(\mu) \langle n \rangle) \\ \forall \lambda, \mu \in \Lambda, n \in \mathbb{N}.$$

This gives the first equality in Conjecture II, by passing to Ext-groups and A, A' . The second and third equalities can be proved similarly, using relevant similar versions of (7.0.30).

Conjecture III is suggested by using ext-groups in an affine Lie algebra setting, assuming Koszulity. It would then follow from the existence there of well-behaved graded translation functors, passing from p -regular to p -singular graded module categories.

(b) One cumulative effect of all the conjectures is to explicitly compute Ext-groups for A and $\tilde{\text{gr}} A$ between objects $\Delta^0(\lambda)$ and $\nabla_0(\mu)$. One can ask if there is any resulting impact on calculations of similar Ext-groups between irreducible modules. We speculate that the homological algebra of $\tilde{\text{gr}} A$, e. g., Ext between irreducible modules or between irreducible

and standard/costandard modules can often be understood in terms of the similar homological algebra of $(\widetilde{\text{gr}} A)_0$, together with formulas like (4.0.20). Observe that $(\widetilde{\text{gr}} A)_0$ is a quotient of A by a nilpotent ideal, so that A , $(\widetilde{\text{gr}} A)_0$, and $\widetilde{\text{gr}} A$ all share the same irreducible modules.

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